Inferring Social Networks from Outbreaks

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How Do We Learn Social Networks?
A social network consists of nodes (agents) and edges (connections). The goal is to determine the structure of a target network.

Active Learning – activate some nodes in the network and observe the effects.
- eg. [Angluin, Aspnes, and Reyzin ’08]
- Often requires the learner having a lot of power.

Passive Learning – observe the network from the outside and make conclusions about its structure.
- This work lies here.
NIHI

2009 Swine Flu – Initial Cases
The Constraints

• The social network is an unknown graph, where nodes are agents.
  ◦ The vertices are known to the learner.
  ◦ The edges are to be learned or estimated.

• Let $p_{(u,v)}$ be the a priori probability of an edge between nodes $u$ and $v$.

• Each observed outbreak induces (or exposes) a constraint.
  ◦ Namely the graph is connected on the induced subset.
Finding the Cheapest Network

- If the prior distribution is independent (and probabilities are small), the maximum likelihood social network maximizes

\[ \prod_{u,v \in V} p(u,v) \]

- This is equivalent to minimizing the sum of the log-likelihood costs

\[ \sum_{v,u \in V} \log(p(u,v)) \]

while satisfying the connectivity constraints.
Finding the Cheapest Network Consistent with the Constraints
Finding the Cheapest Network Consistent with the Constraints
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The Network Inference Problem

• **Given:**
  ◦ vertices $V = \{v_1, \ldots, v_n\}$
  ◦ costs $c_e$ for each edge $e = \{v_i, v_j\}$
  ◦ a constraint set $S = \{S_1, \ldots, S_r\}$, with $S_i \subseteq V$

• **Find**: a set $E$ of edges of lowest cost such that each $S_i$ induces a connected subgraph of $G = (V, E)$
Problem Variants

- We consider both the offline and online version of this problem. We also consider the arbitrary and uniform cost versions.

- Solved for the case where all constraints can be satisfied by a tree [Korach & Stern ’03] – they left the general case open.
  - Our problem is closely related to many network optimization problems.
An Offline Lower Bound

- **Theorem:** If P ≠ NP, the approximation ratio for the Uniform Cost Network Inference problem is \( \Omega(\log n) \).

- **Proof (reduction from Hitting Set)**
  - \( U = \{v_1, v_2, \ldots, v_n\} \)
  - \( C = \{C_1, C_2, \ldots, C_j\} \), with \( C_i \subseteq U \)
  - The Hitting Set problem is to minimize \( |H| \), where \( H \subseteq U \) s.t. \( \forall C_i \) \( H \cap C_i \neq \emptyset \)
An Offline L.B. continued

- Reduction from Hitting Set
- For a constant $k$, we make a $N.I.$ instance

\[ n^k \]

\[ n \]
An Offline L.B. continued

- Reduction from Hitting Set
- For a constant $k$, we make a N.I. instance

Each row corresponds to the elements in the Hitting Set instance.
Constraints: first, for each row, give all pairwise constraints:
An Offline L.B. continued

- Constraints: first, for each row, give all pairwise constraints:

- This will force the learner to put down a clique on each row
An Offline L.B. continued

- Now we have $n^k$ rows of cliques
An Offline L.B. continued

- For each pair of rows:
An Offline L.B. continued

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```
1 2 3 4 5 ... k ... n-1 n

. . . . . . . . . . . . .

. . . . . . . . . . . . .

1 2 3 4 5 ... k ... n-1 n
```
An Offline L.B. continued

- For each pair of rows:

  \[
  \begin{array}{ccccccccc}
  1 & 2 & 3 & 4 & 5 & \ldots & k & \ldots & n-1 & n \\
  \bullet & \bullet & \bullet & \bullet & \bullet & \ldots & \cdots & \cdots & \bullet & \bullet \\
  \bullet & \bullet & \bullet & \bullet & \bullet & \cdots & \cdots & \cdots & \bullet & \bullet \\
  \end{array}
  \]

- w.l.o.g. for the Hitting Set constraint
  - \( C_i = \{v_1, v_2, \ldots, v_k\} \)
  - we will add the constraint:
An Offline L.B. continued

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An Offline L.B. continued

corresponds to adding $v_1$ to $H$

never better
An Offline L.B. continued

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Finishing the Lower Bound

- Unless P=NP, optimal Hitting Set approximation is $\Omega(\log(n))$ [Feige ’98].
- The optimal algorithm pays:
  $$n^k \binom{n}{2} + \text{OPT} \binom{n^k}{2}$$
- But the learner pays:
  $$n^k \binom{n}{2} + \Omega \left( \log(n) \text{OPT} \binom{n^k}{2} \right)$$
- $k$ can be chosen to be sufficiently large.
**Offline Network Inference Algorithm**

- **Theorem:** There is a $O(\log(r))$ approximation algorithm to OPT

- **Algorithm:**
  - Let $C(E)$ sum over all constraints $S_i$, $1$ minus the number of components $S_i$ induces in $G(V,E)$.
  - Consider the greedy algorithm: while $C(E) < 0$, add to $E$ the edge that has the lowest ratio of $c_e$ to $\Delta C$.
  - Note that at the completion of the algorithm, the constraints are all satisfied.
Algorithm Analysis

- The greedy algorithm: while $C(E) < 0$, add to $E$ the edge that has the lowest ratio of $c_e$ to $\Delta C$.

- $C(E)$ is submodular in its edge set.

- A Greedy algorithm for maximizing an integer-valued submodular function on $x$ gives an $H(m)$ approx, where $m = \max_x f(\{x\})$ [Wolsey '82]

- Each edge can increase $C$ by at most $r$, so $m < r$. 
The Online Problem

- Constraints $S_i$ come in online
- Must satisfy each constraint as it comes in.
- Can add but not remove edges.
- Seemingly good ideas like placing an MST on each constraint can perform very badly.
Online Algorithm Against Oblivious Adversary

$O(n^{2/3} \log^{2/3} n)$-competitive algorithm for the uniform cost problem.
Online Algorithm Against Oblivious Adversary

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\[ p = \frac{c \log^{2/3} n}{n^{1/3}} \]
Online Algorithm Against Oblivious Adversary

$O(n^{2/3}\log^{2/3}n)$-competitive algorithm

- All constraints $S_i$, $|S_i| \geq n^{1/3}\log^{1/3}(n)$ are almost surely connected.
- All constraints $S_i$, $|S_i| < n^{1/3}\log^{1/3}(n)$ that are not already covered, we can put a clique on, and hit at least 1 edge in OPT.
- We used $O(n^{5/3}\log^{2/3}(n)+n^{2/3}\log^{2/3}(n)OPT)$ edges in expectation.
- Because $OPT = \Omega(n)$, we are done.
Other Online Results

- The **uniform cost problem** has a $\Omega(\sqrt{n})$-competitive lower bound.

- The competitive ratio for **uniform cost stars** and **paths** is $\theta(\log n)$.
  - for paths, we use pq-trees [Booth and Lueker ’76]

- The **arbitrary cost problem** has an $\Omega(n)$-competitive lower bound and $O(n \log n)$-competitive algorithm.
# Online Results: Competitive Ratios for Adaptive Adversaries

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<th>any G</th>
<th>trees</th>
<th>stars</th>
<th>paths</th>
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<tbody>
<tr>
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<td>$O(n)$</td>
<td>$\Omega(n^{1/2})$</td>
<td>$\Omega(\log n)$</td>
<td>$\Theta(\log n)$</td>
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<tr>
<td>arbitrary cost</td>
<td>$O(n\log n)$</td>
<td>$\Omega(n)$</td>
<td>$O(n\log n)$</td>
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Summary

- New model for passively learning social networks.
  - Also relevant to network optimization..

- Many interesting results.
  - Solve open problem from network optimization.

- Lots of good open problems left!