New Algorithms for Contextual Bandits

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Serving Content to Users

Query, IP address, browser properties, etc.
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result (ie. ad, news story)
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click or not
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context $x_t$

action $j_t$

reward $r_{ir}(t)$
Outline

• Formally define the setting.
• Show ideas that fail.
• Give a high probability optimal algorithm.
• Dealing with VC sets.
• Linear Payoff Functions.
• Slates.
The Setting

- T rounds, K possible actions, N policies $\pi$ in $\Pi$ (context $\rightarrow$ actions)

- for $t=1$ to $T$
  - world commits to rewards $r(t)=r_1(t), r_2(t), \ldots, r_K(t)$
  - world provides context $x_t$
  - learner’s policies recommend $\pi_1(x_t), \pi_2(x_t), \ldots, \pi_N(x_t)$
  - learner chooses action $j_t$
  - learner receives reward $r_{j_t}(t)$

- want to compete with following the best policy in hindsight
Regret

- **reward** of algorithm A:  
  \[ G_A(T) = \sum_{t=1}^{T} r_{j_t}(t) \]

- **expected reward** of policy i:  
  \[ G_i(T) = \sum_{t=1}^{T} \pi_i(x_t) \cdot r(t) \]

- **algorithm A’s regret**:  
  \[ \max_i G_i(T) - G_A(T) \]
Regret

- Algorithm A’s regret: \( \max_i G_i(T) - G_A(T) \)
- Expected regret: \( \max_i G_i(T) - E[G_A(T)] \)
- High probability regret: \( P[\max_i G_i(T) - G_A(T) > \varepsilon] \leq \delta \)
Some Observations

• This is harder than supervised learning. In our setting we do not know the rewards of actions not taken.

• This is not the traditional K-armed bandit setting. In the traditional bandit setting there is no context (or experts).
  • In the simpler K-armed bandit setting, there is no context. We just compete with best arm in hindsight.
  • The traditional setting is akin to showing everyone the same advertisement, article, etc.
## Previous Results

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$\Omega(\sqrt{KT})$ lower bound  [ACFS ’02]
## Our Result

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$\Omega(\sqrt{KT})$ lower bound [ACFS ’02]
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• Formally define the setting.

• **Show ideas that fail.**

• Give a high probability optimal algorithm.

• Dealing with VC sets.

• Linear Payoff Functions.

• Slates.
First Some Failed Approaches

• **Bad idea 1**: Maintain a set of plausible hypotheses and randomize uniformly over their predicted actions.

Adversary has two actions, one always paying off 1 and the other 0. Hypothesis generally agree on correct action, except for a different one which defects each round. This incurs regret of \(\frac{T}{2}\).

• **Bad idea 2**: Maintain a set of plausible hypotheses and randomize uniformly among the hypothesis.

Adversary has two actions, one always paying off 1 and the other 0. If all but one of \(2T\) hypothesis always predict wrong arm, and only 1 hypothesis always predicts good arm, with probability \(\frac{1}{2}\) it is never picked and algorithm incurs regret of \(T\).
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epsilon-greedy

- Rough idea of $\varepsilon$-greedy (or $\varepsilon$-first): act randomly for $\varepsilon$ rounds, then go with best (arm or expert).

- Even if we know the number of rounds in advance, epsilon-first won’t get us regret $O(T)^{1/2}$, even in the non-contextual setting.

- Rough analysis: even for just 2 arms, we suffer regret: $\varepsilon + (T-\varepsilon)/(\varepsilon^{1/2})$.
  - $\varepsilon = T^{2/3}$ is optimal.
  - gives regret $T^{2/3}$
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Ideas Behind Exp4.P

- **exponential weights**
  - keep a weight on each expert that drops exponentially in the expert’s (estimated) performance

- **upper confidence bounds**
  - use an upper confidence bound on each expert’s estimated reward

- **ensuring exploration**
  - make sure each action is taken with some minimum probability

- **importance weighting**
  - give rare events more importance to keep estimates unbiased
Exponential Weight Algorithm for Exploration and Exploitation with Experts

(EXP4) [Auer et al. ’95]

Initialization: \( \forall \pi \in \Pi : w_t(\pi) = 1 \)

For each \( t = 1, 2, \ldots \):

1. Observe \( x_t \) and let for \( a = 1, \ldots, K \)

\[
p_t(a) = (1 - Kp_{\text{min}}) \frac{\sum \pi \mathbf{1}[\pi(x_t) = a] w_t(\pi)}{\sum \pi w_t(\pi)} + p_{\text{min}},
\]

where \( p_{\text{min}} = \sqrt{\frac{\ln |\Pi|}{KT}} \).

2. Draw \( a_t \) from \( p_t \), and observe reward \( r_t(a_t) \).

3. Update for each \( \pi \in \Pi \)

\[
w_{t+1}(\pi) = \begin{cases} w_t(\pi) \exp \left( p_{\text{min}} \frac{r_t(a_t)}{p_t(a_t)} \right) & \text{if } \pi(x_t) = a_t \\ w_t(\pi) & \text{otherwise} \end{cases}
\]
Exponential Weight Algorithm for Exploration and Exploitation with Experts

(Exp4.P) [Beygelzimer, Langford, Li, R, Schapire ’10]

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   where \( p_{\text{min}} = \sqrt{\frac{\ln |\Pi|}{KT}} \).

2. Draw \( a_t \) from \( p_t \), and observe reward \( r_t(a_t) \).

3. Update for each \( \pi \in \Pi \)
   \[
   w_{t+1}(\pi) = w_t(\pi) \exp \left( \frac{p_{\text{min}}}{2} \left( 1[\pi(x_t) = a_t] \frac{r_t(a_t)}{p_t(a_t)} + \frac{1}{p_t(\pi(x_t))} \sqrt{\frac{\ln N/\delta}{KT}} \right) \right)
   \]
Why Should This Work?

\[ w_i(t + 1) = w_i(t) \exp \left( \frac{\sqrt{\ln N}}{2\sqrt{KT}} \left( \hat{y}_i(t) + \hat{v}_i(t) \sqrt{\frac{\ln(N/\delta)}{KT}} \right) \right) \]
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\[ \sim \sum_{i} \hat{y}_i(t) + \sqrt{\ln(N/\delta)/KT} \sum_{i} \hat{v}_i(t) \]

\[ \sim \hat{G}_i + \sqrt{\ln(N/\delta)} \hat{\sigma}_i \]

\[ \geq G_i \text{ w.p. } \geq 1 - \delta \]

using a martingale inequality
Proof Outline

so we have

\[ \hat{G}_i + \sqrt{\ln(N/\delta)} \hat{\sigma}_i \geq G_i \text{ w.p. } \geq 1 - \delta \]
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letting

\[ \hat{U} = \max_i \left( \hat{G}_i + \sqrt{\ln(N/\delta)}\hat{\sigma}_i \right) \]

by looking at

\[ \ln(W_{T+1}/W_1) \]

we can show

\[ G_{\text{Exp4.P}} \geq (1 - 2\sqrt{K \ln N / T})\hat{U} - \ln(N/\delta) \]

\[ -2\sqrt{KT\ln N} - \sqrt{KT\ln(N/\delta)} \]
so we have

\[ \hat{G}_i + \sqrt{\ln(N/\delta)}\hat{\sigma}_i \geq G_i \text{ w.p. } \geq 1 - \delta \]

and

\[ G_{\text{Exp4.P}} \geq (1 - 2\sqrt{K\ln N/T})\hat{U} - \ln(N/\delta) - 2\sqrt{KT\ln N} - \sqrt{KT\ln(N/\delta)} \]
Proof Outline

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\[ -2\sqrt{KT\ln N} - \sqrt{KT\ln(N/\delta)} \]

implies
\[ G_{\text{Exp4.P}} \geq G_{\text{max}} - O\left(\sqrt{KT\ln(N/\delta)}\right) \text{ w.p. } 1 - \delta \]

Exp4P beats epsilon-greedy in practice [BLLRS ’10] and performs negligibly worse (on average) than Exp4.
One Problem…

- This algorithm requires keeping explicit weights on the policies.
  - Okay for polynomially many policies.
  - Okay for some special cases.
  - Not efficient in general.

- Want an efficient algorithm that would (for example) work with an ERM Oracle
  - epoch-greedy [Langford and Zhang ’07] has this property.
## Results

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Infinitely Many Policies

• What if we have an infinite number of policies?
• Our bound of $\tilde{O}(K \ln(N)T)^{1/2}$ becomes vacuous.
• If we assume our policy class has a finite VC dimension $d$, then we can tackle this problem.
• Need i.i.d. assumption. We will also assume $k=2$ to illustrate the argument.
VC Dimension

- The VC dimension of a hypothesis class captures the class’s expressive power.
- It is the cardinality of the largest set (in our case, of contexts) the class can shatter.
  - Shatter means to label in all possible configurations.
VE, an Algorithm for VC Sets

The VE algorithm [Beygelzimer, Langford, Li, R, Schapire ’10] :

• Act uniformly at random for $\tau$ rounds.

• This partitions our policies $\Pi$ into equivalence classes according to their labelings of the first $\tau$ examples.

• Pick one representative from each equivalence class to make $\Pi'$.

• Run Exp4.P on $\Pi'$. 
Outline of Analysis of VE

• Sauer’s lemma bounds the number of equivalence classes to \((e\tau/d)^d\).
  • Hence, using Exp4.P bounds, VE’s regret to \(\Pi’\) is \(\approx \tau + O(Td \ln(\tau))\)

• We can show that the regret of \(\Pi’\) to \(\Pi\) is \(\approx (T/\tau)(d \ln T)\)
  • by looking at the probability of disagreeing on future data given agreement for \(\tau\) steps.

• \(\tau \approx (Td \ln 1/\delta)^{1/2}\) achieves the optimal trade-off.

• **Gives \(\tilde{O}(Td)^{1/2}\) regret.**

• Still inefficient!
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- **Linear Payoff Functions.**
- Slates.
When Do Efficient Algorithms Exist?

- One example is the **linear payoff functions** setting.

- In this setting, on each round, we observe a \( d \) dimensional context (feature vector) for each arm.
  - Equivalent to the contextual bandit setting.

- We assume there exists an unknown vector, whose dot product with each arm feature gives the expected regret of that arm.
  - Similar to a realizability assumption.
Linear Payoffs

- LinRel [Auer ’02] gives a polynomial time algorithm with $\tilde{O}(Td)^{1/2}$ regret.
- LinRel tries to estimate the reward of the current round by looking at past rounds.
  - LinRel decomposes the feature vector of the current round into a linear combination of feature vectors seen on previous rounds.
  - Looks at previous rewards to compute coefficients.
  - Uses these estimates to compute reward estimates.
- Matches the $\Omega(Td)^{1/2}$ lower bound [Chu, Li, R, Schapire ’10] up to log factors.
- LinUCB (a similar algorithm to LinRel) outperforms epsilon-greedy on Yahoo! Homepage data. [Li, Chu, Langford, Schapire ’10].
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Slates
Oftentimes, we have to choose multiple actions (without replacement).

Different models:
- All slots equal.
- Positional factors.
- Interaction via “properties.”
Slates

- For $T$ time steps, $K$ actions, $s$ slots, $N$ experts choosing slates, we can get the following regret bounds [Kale, R, Schapire ’10]:
  - $O((sKT\ln N)^{1/2})$ for unordered slates
  - $O(s(KT\ln N)^{1/2})$ for ordered slates

- Beats a straightforward reduction to Exp4.
- Uses a variant of multiplicative weights.
- Not efficient.
Summary

• Described Exp4P, the first optimal high probability algorithm for the contextual bandit problem.

• Showed how to compete with a VC-Set.

• Discussed an efficient linear-payoff algorithm.

• Introduced the slates problem.
Main Open Problem: Find an efficient optimal algorithm for the contextual bandits problem!

i.e. make the Exp4P algorithm efficient with an ERM (Empirical Risk Minimization) oracle.

• Instead of updating weights explicitly for each policy, feed rewards for actions into an oracle, which can return a good policy.
• This oracle could be a standard efficient learning algorithm.
Open Problems

• Find good classes of policies for contextual bandit problems. Linear policies seem to do well…

• Get rid of the realizability assumption in LinRel or LinUCB for linear payoffs.

• Deal with interaction in slates!

• More experimental evaluation.