On the Resilience of Bipartite Networks

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work joint with Shelby Heinecke and Will Perkins
(Appeared in Allerton 2018)

Model

μ: probability of being infected "by nature"

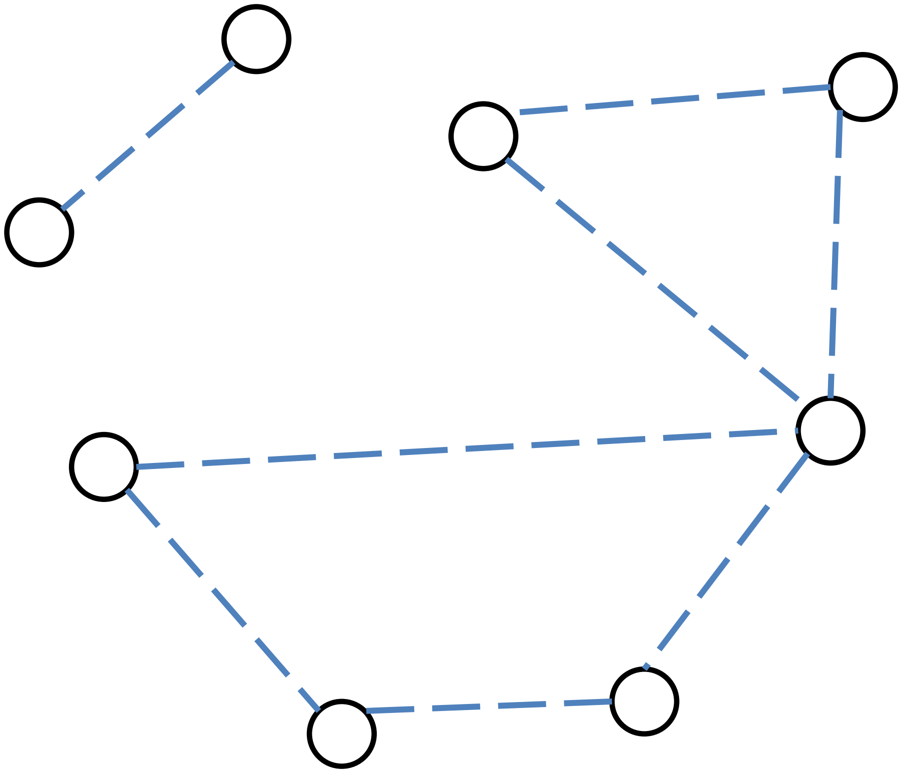
p: probability of passing infection to neighbor

$G = (V,E)$: a graph

Independent cascade infections:

- each infected node gets 1 independent chance of passing infection to neighbors in G.

Example



μ = 0.25

p = 0.5

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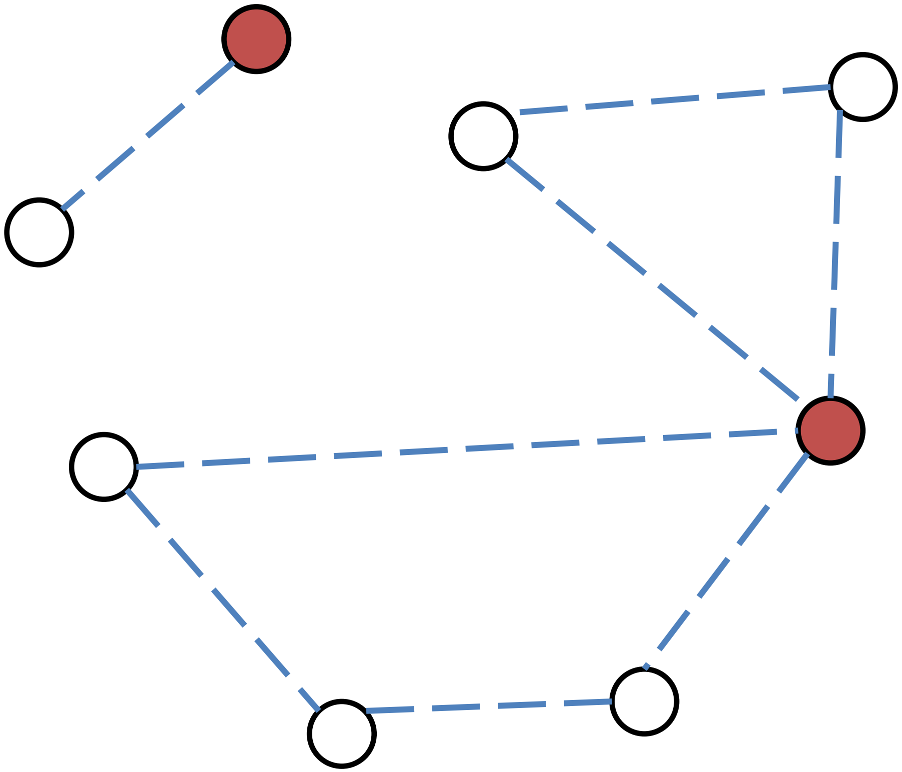

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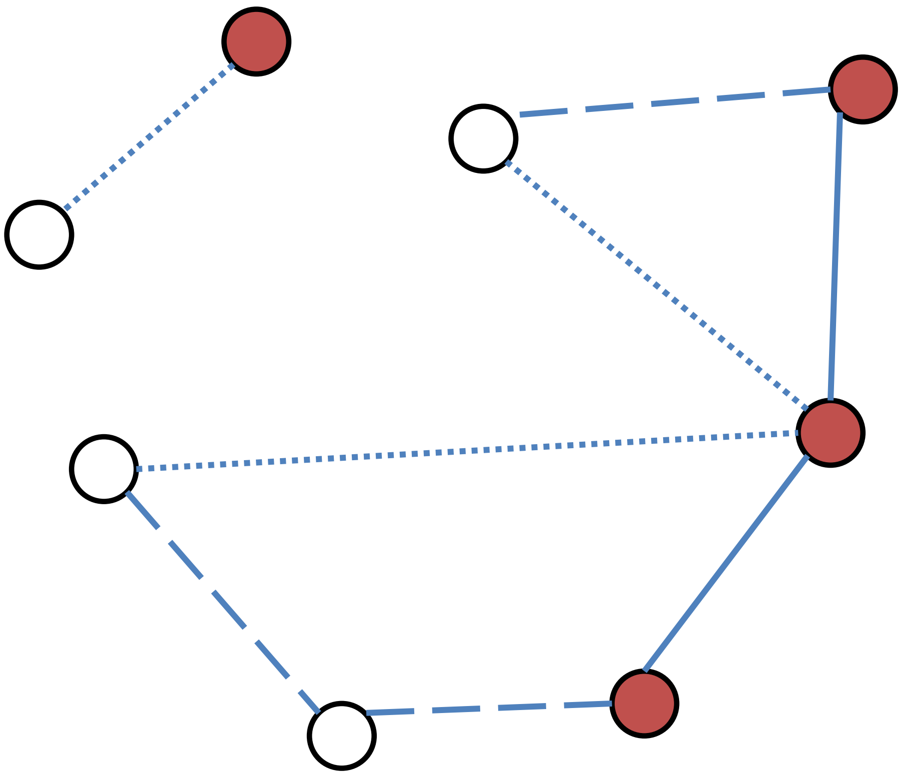

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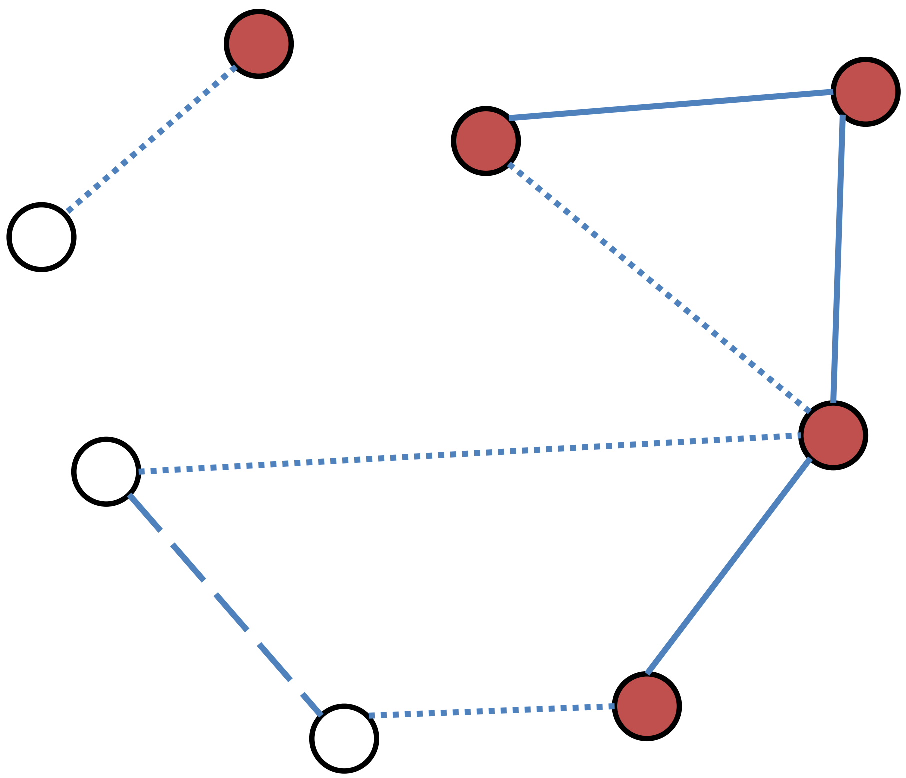

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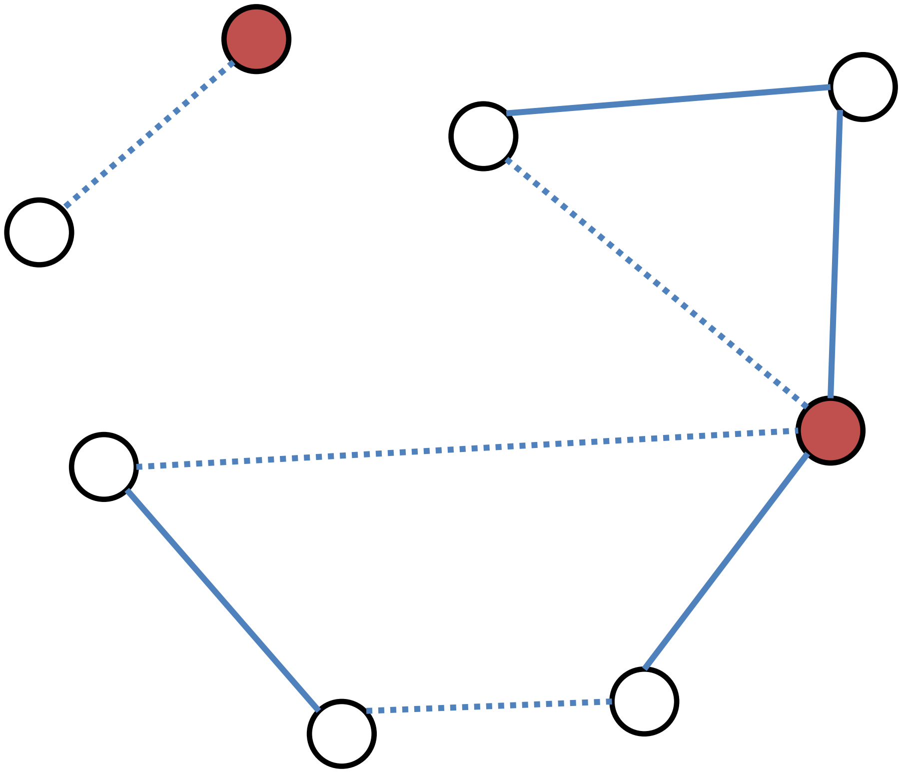
Equivalently, Percolation



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After Edge Percolation

Let |C(v)| be the size of the random connected component containing vertex v.

It is easy to calculate that

$$I(G) := 1 - \frac{1}{n}\mathbb{E}\left[\sum_{v \in G} (1-\mu)^{|C(v)|}\right]$$

is the expected fraction of infected nodes.

Susceptibility

Fundamental quantity in the study of random graphs called susceptibility:

$$S(G) = \frac{1}{n} \sum_{v} |C(v)|$$

Observation: minimizing E(S(G)) after percolation minimizes expected number of infections in a "single-origin" infection model.

Main Question

Which networks are most "resilient"?
i.e. given μ and p, which edge structure will produce the smallest expected number infected.

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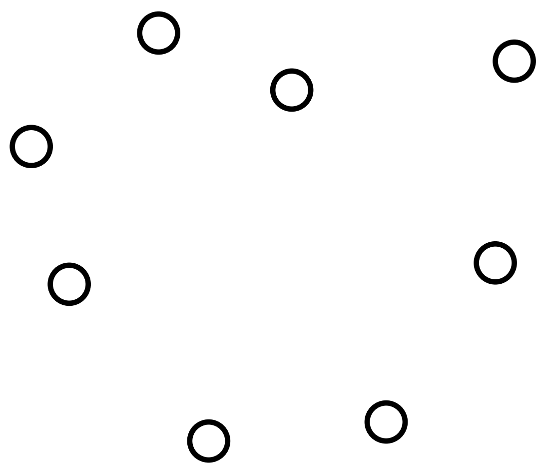
Clearly G=(V,Ø)



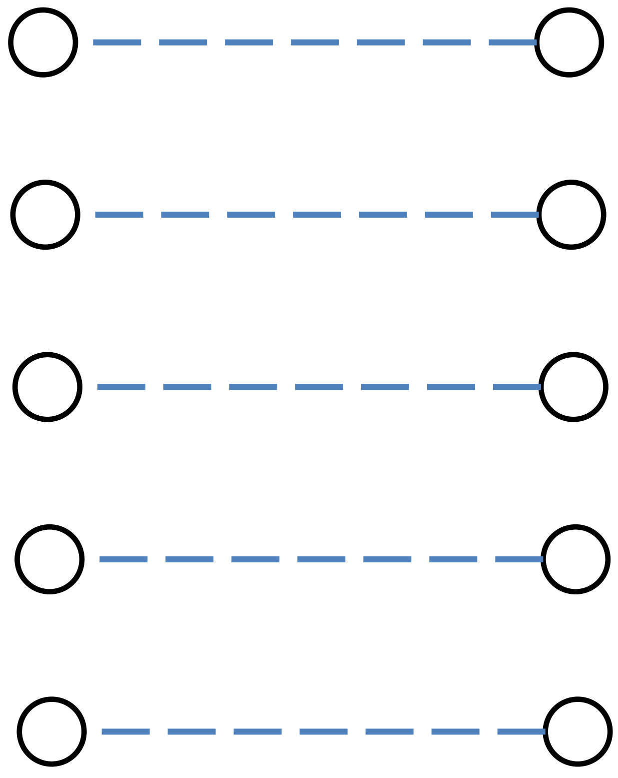
Main Question

Which networks of min degree d are most “resilient”? I.e. given μ and p, which edge structure will produce the smallest expected number infected.

Studied by Blume-Easley-Kleinberg-Kleinberg-Tardos (FOCS ’11)

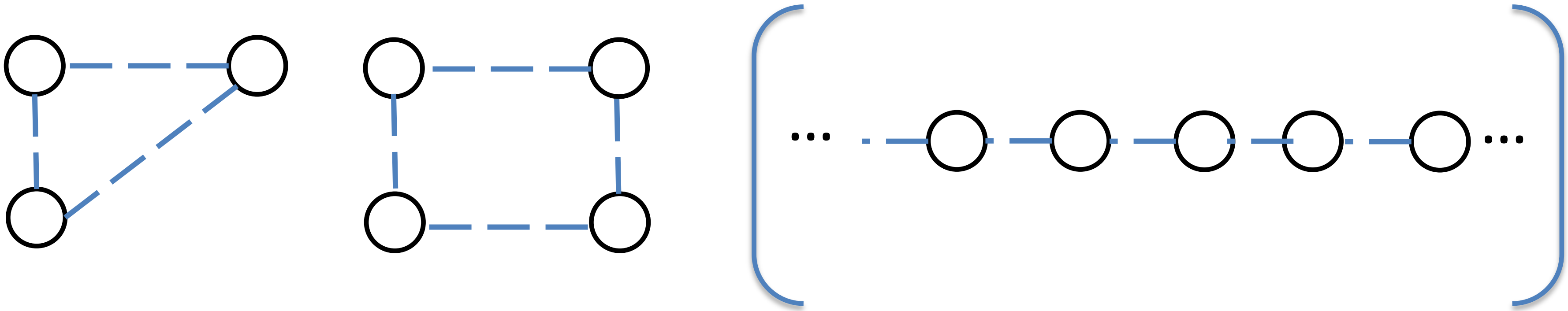
d=1

trivial:



d=2

Cycle decomposition (or perhaps infinite path?):



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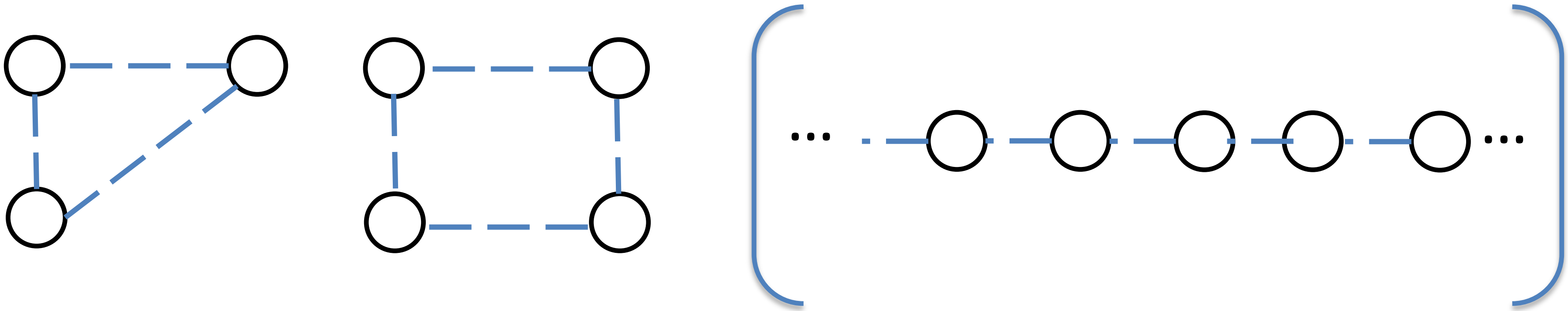


Theorem: Blume-Easley-Kleinberg-Kleinberg-Tardos (11)

Smaller cycles are always better: optimal is always a triangle decomposition (or an infinite path)

d>2

Blume et al ('11) show:

At least 3 different graphs can be optimal, depending on settings μ,p.

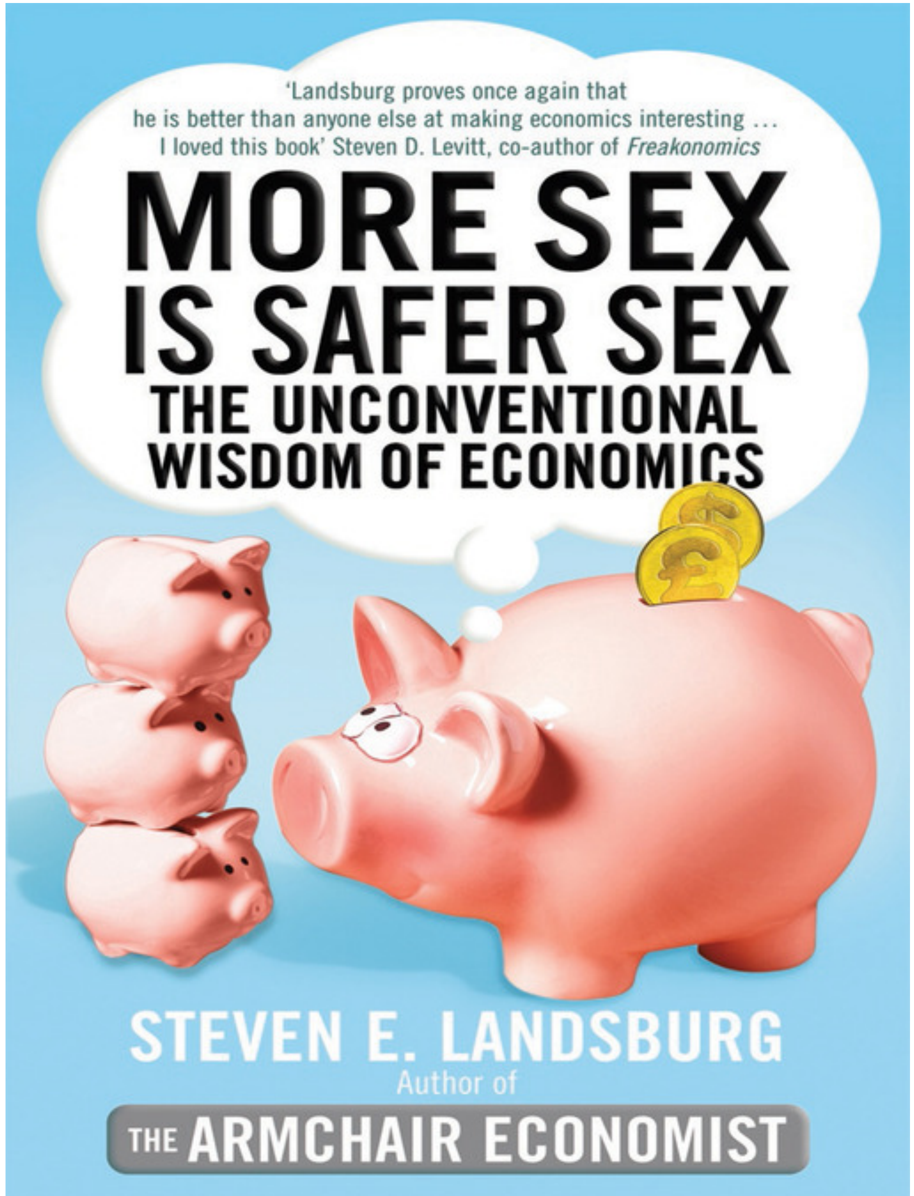
But, not completely characterized! This problem quickly (and surprisingly) gets hard.

Bipartite Networks

Kremer ('95) showed in a model of STD spread in heterosexual contact networks, two extreme equilibria can occur, roughly:

- everyone has same number of partners
- some individuals have very many partners, and most have significantly fewer.

In part of his paper, he assumes preferences differ between genders. Men roughly have same activity, women are allowed to vary.


'Landsburg proves once again that
he is better than anyone else at making economics interesting ...
I loved this book' Steven D. Levitt, co-author of *Freakonomics*
MORE SEX
IS SAFER SEX
THE UNCONVENTIONAL
WISDOM OF ECONOMICS
STEVEN E. LANDSBURG
Author of
THE ARMCHAIR ECONOMIST

Independent Cascade

This presents a natural question in the independent cascade model in half-regular bipartite graphs.

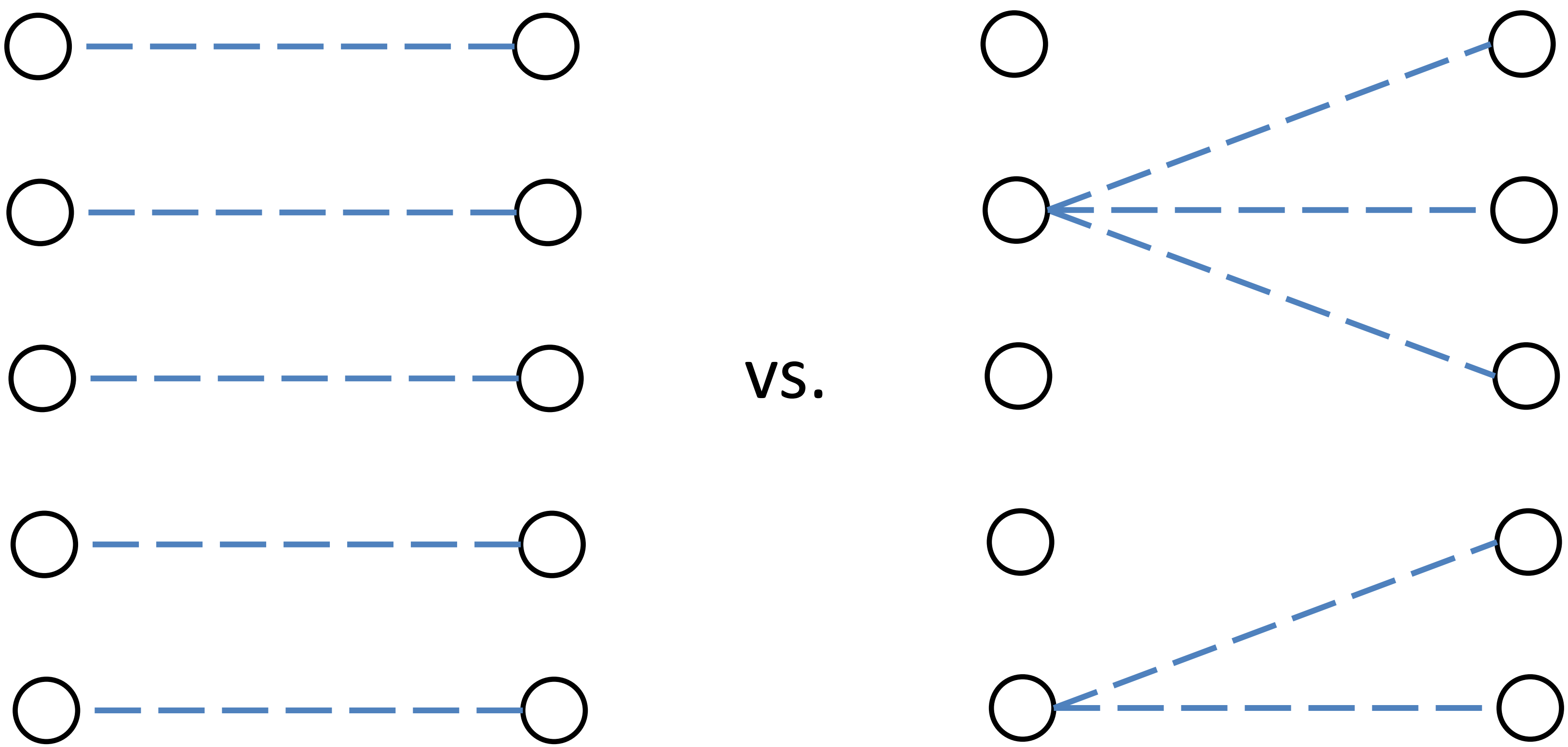
problem: given a bipartite graph on 2n vertices,

V = {L,R}, a degree-restriction d for one side of the bipartition (deg(R)>d), and μ and p, what is the most resilient network?

(overoveroversimplified for accurately modeling real-world settings)

Natural model for other domains (clients/servers persons/drinking wells).

d=1, no longer trivial



$d=1$

Theorem: for all values of μ and p, either a matching or a star (plus isolated vertices) is optimal.

In fact, for $\mu \leq ½$ a matching is optimal. Otherwise a star is optimal.

Analysis (d=1)

L_k = prob a degree k node in L is infected and
R_k = prob a node in R joined to a degree k node in L is infected
overall probability is convex combination of stars

So, in a k-star with isolated vertices:

$$E[I_k] = (L_k + (k-1)L_0 + kR_k)/2k$$

and

$$L_j = 1 – (1-\mu)(1-\mu p)^j$$
$$R_j = \mu + p – \mu p – (1-\mu)^2(1-\mu p)^{j-1}$$

can show $E[I_k]$ can always be improved unless k = 1 or d

Already, Surprising Behavior

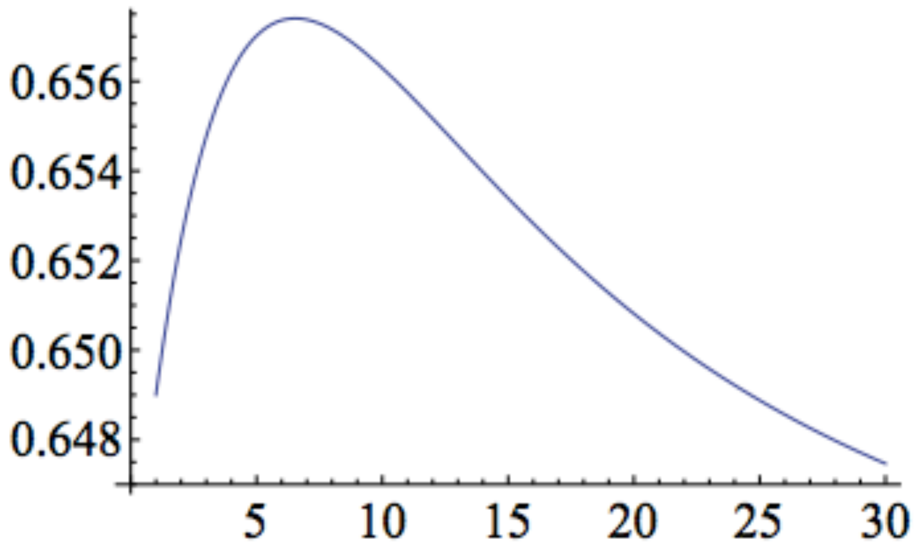
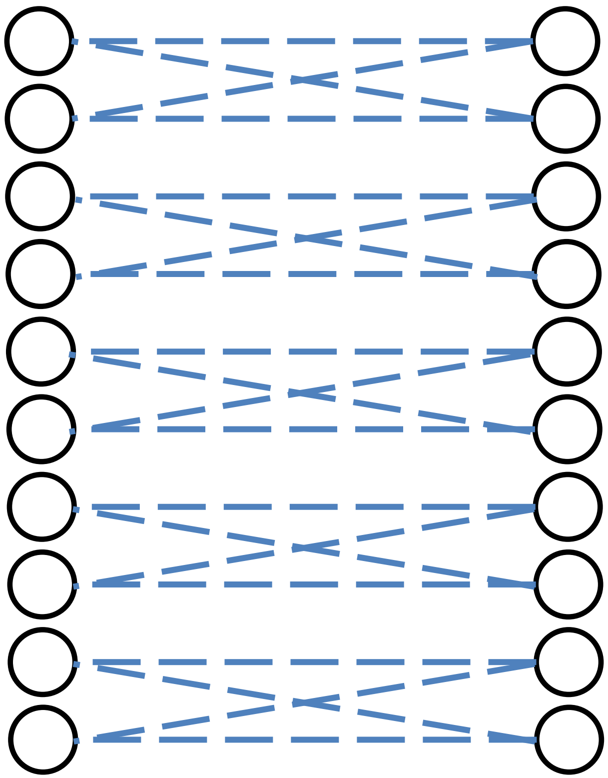


Fig. 1. Average infection probability as a function of the degree of a star, for $\mu = 0.55$ and $p = 0.4$.

How do you explain this intuitively?

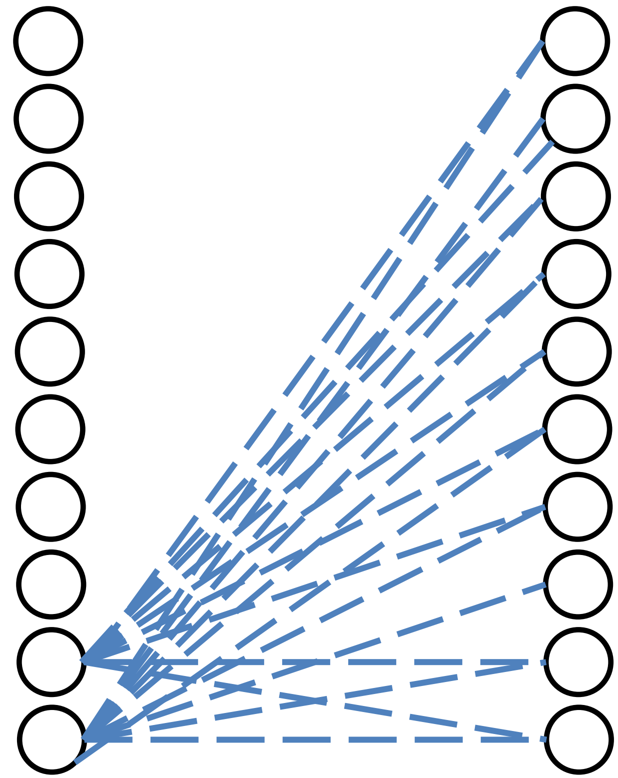
For d>1

Becomes difficult. Natural generalization would be that $K_{d,d}$ decomposition or $K_{d,n}$ is always optimal:



for d=2

$K_{2,2}$ vs. $K_{2,n}$



But for d > 1, the parameter p matters!

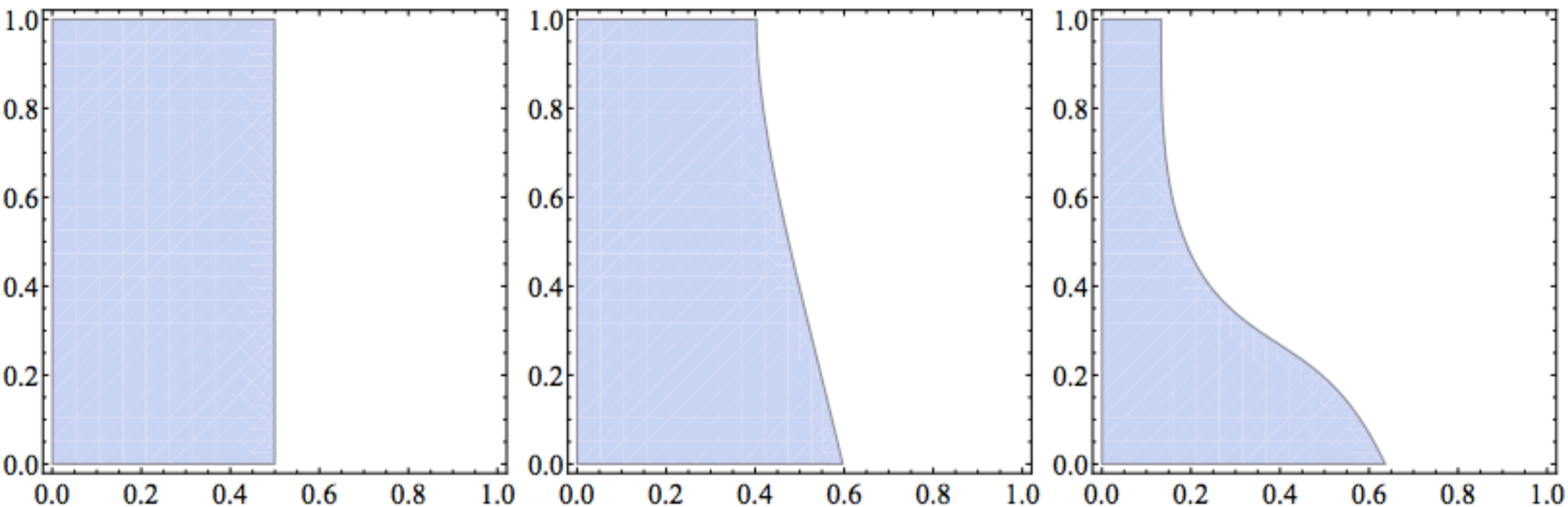


Fig. 2. The graphs are for $d = 1$ (left), $d = 2$ (center), and $d = 3$ (right), for $n \to \infty$. The x-axes are values of μ, and the y-axes are values of p. The colored regions are where a $K_{d,d}$ decomposition has a lower average infection rate than $K_{d,n}$ with $n - d$ isolated vertices.

For d>1

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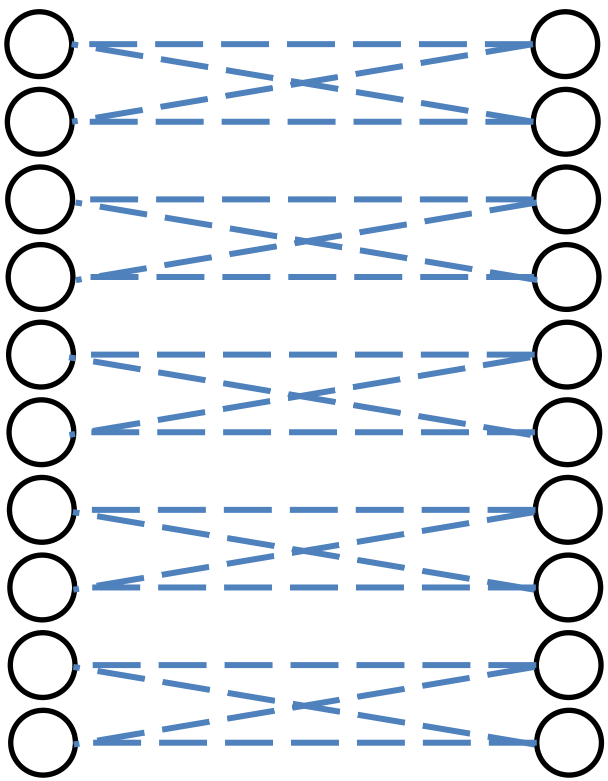
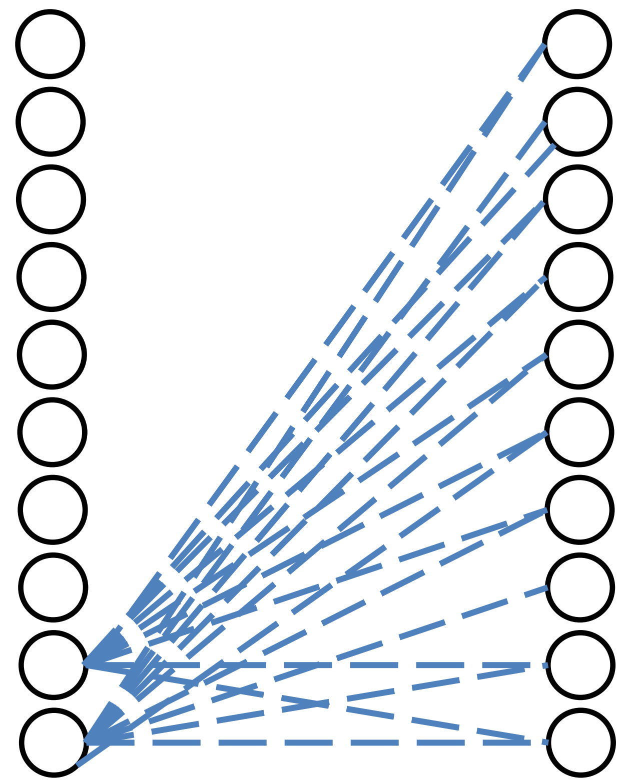


for d=2

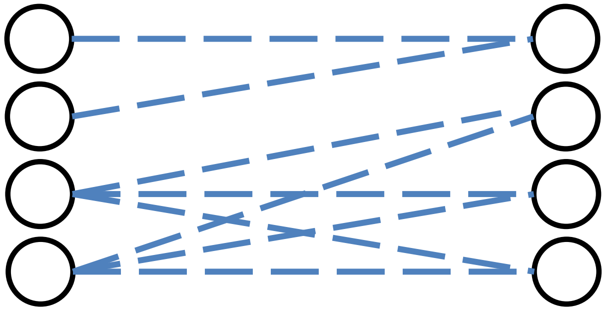
$K_{2,2}$ vs. $K_{2,n}$



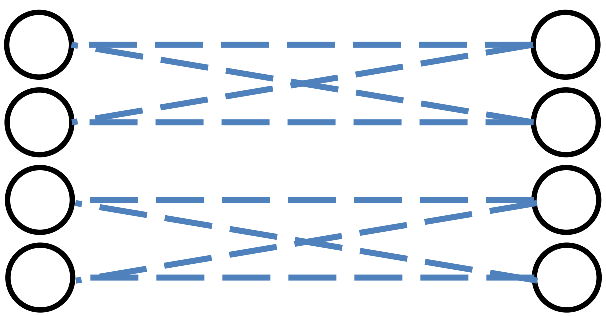
This was our conjecture, but we didn't know how to prove it!

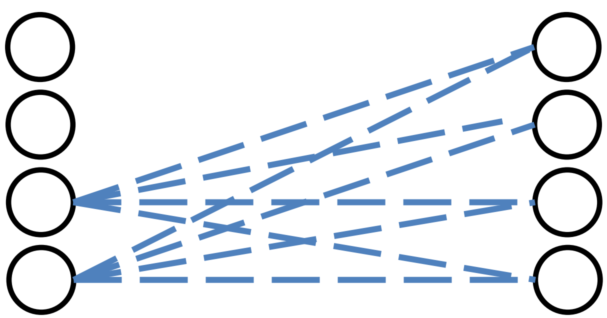
Conjecture is false for d=2!

For μ = .302 and p = .801, the graph below is most resilient.



(It's more resilient than the two conjectured optima, below.)





What we know for d>1

There exist non-trivial settings where a $K_{d,d}$ decomposition and $K_{d,n}$ are optimal:

when $\mu = 1-1/n^2$, need to maximize expected number of isolated vertices after percolation. Can show that this is achieved by $K_{d,n}$.

when $\mu = 1/n^2$, need to minimize average expected component size. This is achieved by $K_{d,d}$ decomposition.

Basic Summary

	regular graphs [Blume et all 11]	“half-regular” bipartite graphs
d = 1	trivial	characterized
d = 2	characterized	extremal results + counterexamples
d > 2	extremal results	extremal results + counterexamples

Connected Graphs

Instead of the half-regular bipartite graphs, one can ask what is most resilient connected graph.

Easy to see that all optima are trees, which are bipartite and have average degree ≈ 2.

Optimal Trees

Natural conjecture: the path graph or star graph is always optimal.

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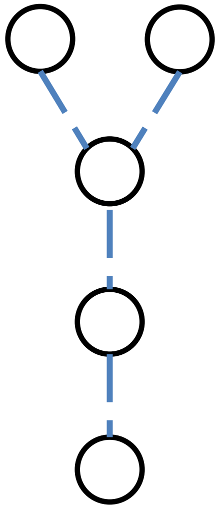
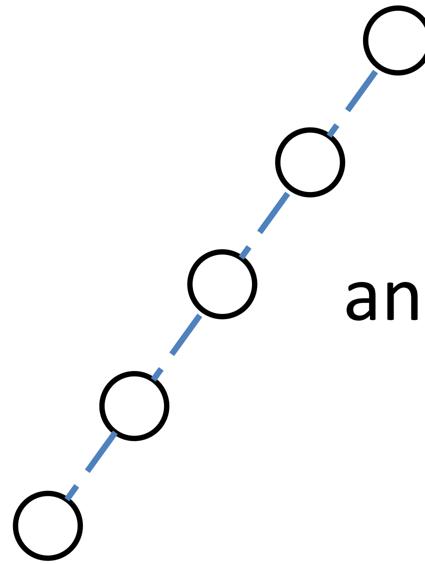
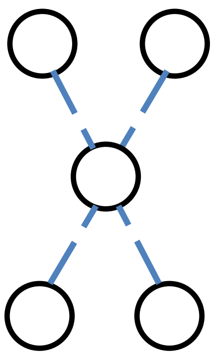
Turns out to be wrong!



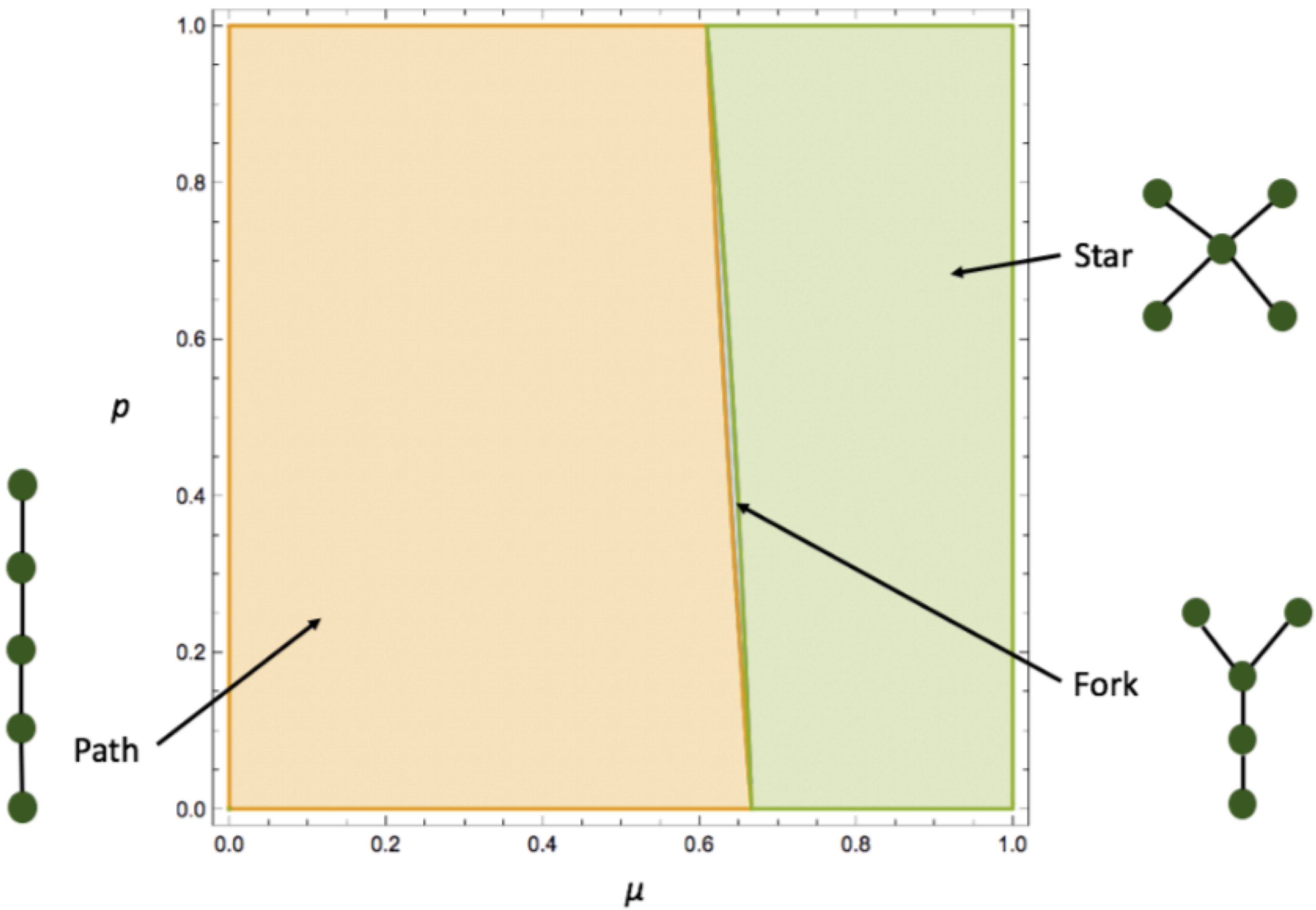
can be more resilient than



and



Comparison



Restricted Version

Consider the same problem when the input is a graph G=(V,E) and the solution is its most resilient half-regular bipartite subgraph.

i.e. not all connections are allowed

Result: this optimization problem is NP-Hard for all $d \geq 1$.

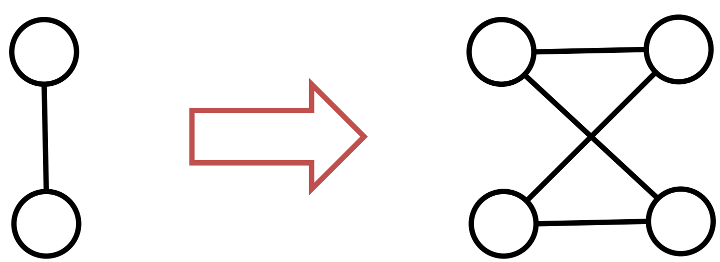
NP-Hardness

<u>d≥3</u>:

Consider a setting where $K_{d,d}$ decomposition is optimal.

Finding a d-clique decomposition is NP-hard for d≥3 for arbitrary graphs (Kirkpatrick-Hell '78)

Take the "double cover" with self-edges



NP-Hardness

d=2:

Even easier: finding a 4-cycle decomposition of a bipartite graph is NP-Hard (Feder-Motwani '95)

So, consider a setting where $K_{2,2}$ decomposition is optimal.

NP-Hardness

d=1:

Use setting where optimal subgraph maximizes number of isolated vertices.

There is a reduction from exact set cover (with the L as the sets and R as the elements).

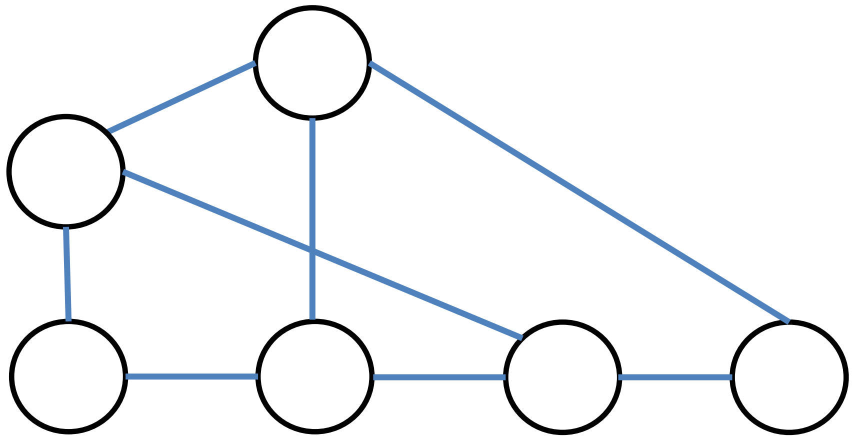
General Threshold Model

Here, each vertex i is assigned an integer threshold $u_i \geq 0$, *i.i.d. from common distribution*.

If $u_i = 0$, then i is infected by nature.

Otherwise, it is infected if and only if u_i of its neighbors are.

General Threshold Model



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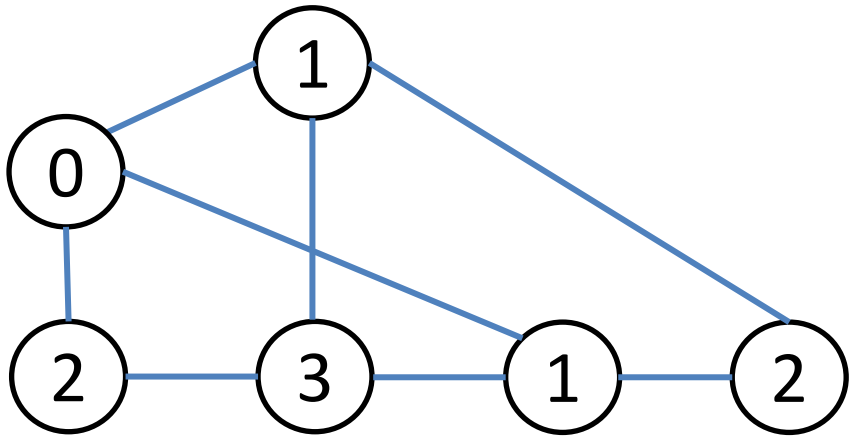

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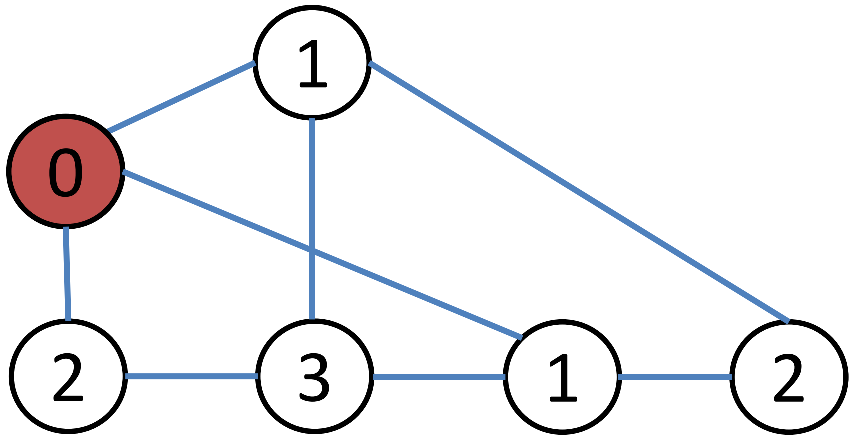

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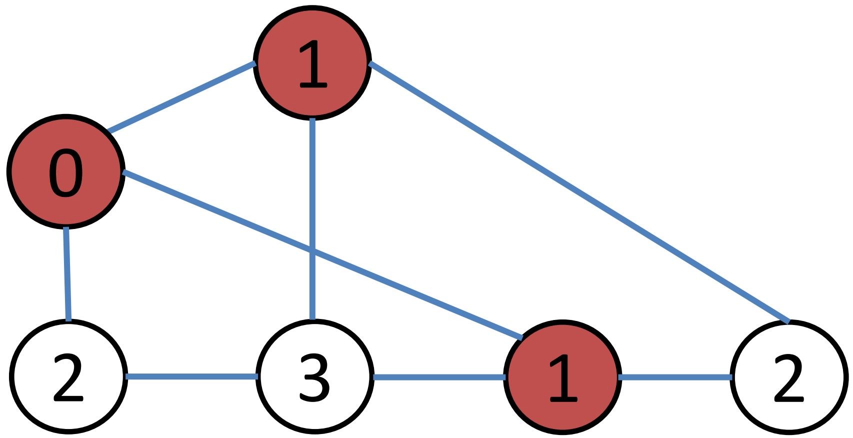

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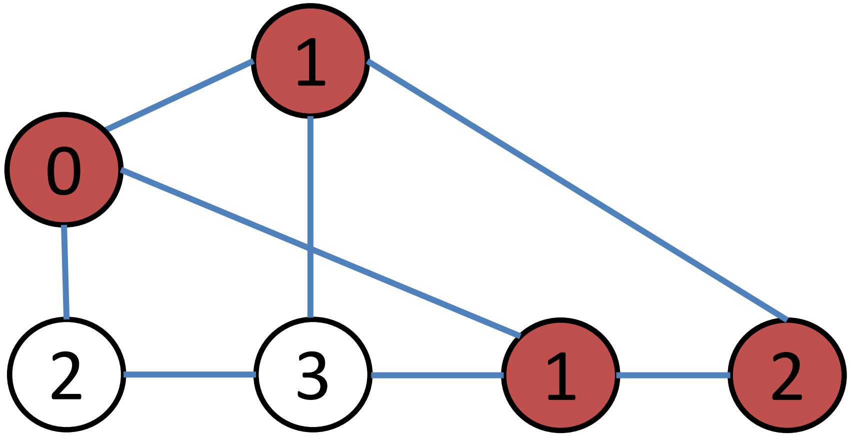

General Threshold Model
More General

μ,p model is a special case with the distribution:

$$\mu_i = \begin{cases} \mu & \text{if } i = 0 \\ \mu_i = (1-\mu)p(1-p)^{i-1} & \text{if } i \geq 1. \end{cases}$$

General Threshold Model
Strictly More General

Theorem: for d=1, for each k ≥1 there is a probability distribution over μ_is such that a k-star decomposition is optimal.

General Threshold Model

<u>Strictly</u> More General

Theorem: for d=1, for each k ≥1 there is a probability distribution over μ_is such that a k-star decomposition is optimal.

Distribution:

Set $u_0 = .6$

$u_1 = \varepsilon$

$u_{k+1} = .4-\varepsilon$

Open Questions

In the independent cascade μ,p model, can we (better) characterize the half-regular bipartite graphs? (& solve Blume et al.'s ('11) open problems.)

Can we characterize which connected graphs optimal?

Can we find approximation algorithms for the NP-hard variants.

What happens in nature?