Aggressive Learning for Contextual Bandits

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Example of Learning through Exploration

Repeatedly:

1. A user comes to Yahoo! (with history of previous visits, IP address, data related to his Yahoo! account)

2. Yahoo! chooses information to present (from urls, ads, news stories)

3. The user reacts to the presented information (clicks on something, clicks, comes back and clicks again, et cetera)

Yahoo! wants to interactively choose content and use the observed feedback to improve future content choices.
Another Example: Clinical Decision Making

Repeatedly:

1. A patient comes to a doctor with symptoms, medical history, test results
2. The doctor chooses a treatment
3. The patient responds to it

The doctor wants a policy for choosing targeted treatments for individual patients.
The Contextual Bandit Setting

For $t = 1, \ldots, T$:

1. The world produces some context $x_t \in X$
2. The learner chooses an action $a_t \in \{1, \ldots, K\}$
3. The world reacts with reward $r_t(a_t) \in [0, 1]$

Goal:
The Contextual Bandit Setting

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**Goal:** Learn a good policy for choosing actions given context.
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What does learning mean?
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What does learning mean? Efficiently competing with a large reference class of possible policies $\Pi = \{\pi : X \rightarrow \{1, \ldots, K\}\}$:

$$\text{Regret} = \max_{\pi \in \Pi} \sum_{t=1}^{T} r_t(\pi(x_t)) - \sum_{t=1}^{T} r_t(a_t)$$

Other names: associative reinforcement learning, associative bandits, learning with partial feedback, bandits with side information
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Basic Observation #1

This is not a supervised learning problem:

- We don’t know the reward of actions not taken—loss function is unknown even at training time.
- Exploration is required to succeed (but still simpler than reinforcement learning – we know which action is responsible for each reward)
Basic Observation #2

This is not just a bandit problem:

- In the bandit setting, there is no $x$, and the goal is to compete with the set of constant actions. Too weak in practice.
- Generalization across $x$ is required to succeed.
One algorithm was known: EXP4

Theorem: [Auer et al. ’95] For all oblivious sequences $(x_1, r_1), \ldots, (x_T, r_T)$, EXP4 has expected regret

$$O\left(\sqrt{TK \ln |\Pi|}\right).$$

Theorem: [Auer et al. ’95] For any $T$, there exists an iid sequence such that the expected regret of any player is $\Omega(\sqrt{TK})$.

EXP4 can be modified to succeed with high probability or over VC sets when the world is IID. [Beygelzimer, et al. 2011].

EXP4 is slow

$$\Omega(T|\Pi|)$$

Exponentially slower than is typical for supervised learning. Can we make an algorithm taking advantage of supervised learning systems?
Policy Elimination

Let \( \Pi_0 = \Pi \)
\( \mu_t = 1/\sqrt{Kt} \)
\( \eta_t(\pi) = \frac{1}{t} \sum_{x,a,r_a} \frac{r_al(\pi(x)=a)}{p(a|x)} \)

For each \( t = 1, 2, \ldots \)

1. Choose distribution \( P_t \) over \( \Pi_{t-1} \) s.t. \( \forall \pi \in \Pi_{t-1}: \)

\[
E_{x \sim D_x} \left[ \frac{1}{(1 - K\mu_t) \Pr_{\pi' \sim P_t}(\pi'(x) = \pi(x)) + \mu_t} \right] \leq 2K
\]

2. observe \( x_t \)

3. Let \( p_t(a) = (1 - K\mu_t) \Pr_{\pi \sim P_t}(\pi(x) = a) + \mu_t \)

4. Choose \( a_t \sim p_t \) and observe reward \( r_t \)

5. Let \( \Pi_t = \{ \pi \in \Pi_{t-1} : \eta_t(\pi) \geq \max_{\pi' \in \Pi_{t-1}} \eta_t(\pi') - K\mu_t \} \)
Policy Elimination

Let $\Pi_0 = \Pi$

$\mu_t = \text{minimum action probability}$

$\eta_t(\pi) = \text{importance weighted empirical reward estimate}$

For each $t = 1, 2, \ldots$

1. Choose distribution $P_t$ over $\Pi_{t-1}$ s.t. $\forall \pi \in \Pi_{t-1}$:

$$E_{x \sim D_x} \left[ \frac{1}{(1 - K\mu_t) \Pr_{\pi' \sim P_t}(\pi'(x) = \pi(x)) + \mu_t} \right] \leq 2K$$

2. observe $x_t$

3. Let $p_t(a) = (1 - K\mu_t) \Pr_{\pi \sim P_t}(\pi(x) = a) + \mu_t$

4. Choose $a_t \sim p_t$ and observe reward $r_t$

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Policy Elimination

Let $\Pi_0 = \Pi$

$\mu_t =$ minimum action probability

$\eta_t(\pi) =$ importance weighted empirical reward estimate

For each $t = 1, 2, \ldots$

1. Find a distribution $P$ over remaining policies $\Pi_{t-1}$ which makes the probability of each remaining policy’s action $> \frac{1}{K}$.

2. Observe $x_t$

3. Let $p_t(a) = (1 - K \mu_t) \Pr_{\pi \sim P_t}(\pi(x) = a) + \mu_t$

4. Choose $a_t \sim p_t$ and observe reward $r_t$

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2. observe \( x_t \)
3. Project the distribution over policies onto a distribution over actions \( p_t \).
4. Choose \( a_t \sim p_t \) and observe reward \( r_t \).
5. Let \( \Pi_t = \{ \pi \in \Pi_{t-1} : \eta_t(\pi) \geq \max_{\pi' \in \Pi_{t-1}} \eta_t(\pi') - K \mu_t \} \)
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Let $\Pi_0 = \Pi$

$\mu_t = \text{minimum action probability}$

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For each $t = 1, 2, \ldots$

1. Find a distribution $P$ over remaining policies $\Pi_{t-1}$ which makes the probability of each remaining policy’s action $> \frac{1}{K}$.
2. observe $x_t$
3. Project the distribution over policies onto a distribution over actions $p_t$.
4. Choose $a_t \sim p_t$ and observe reward $r_t$
5. Let $\Pi_t = \text{those policies not much worse than the empirical best.}$
Analysis

For all sets of policies $\Pi$, for all distributions $D(x, \bar{r})$, if the world is IID w.r.t. $D$, with high probability Policy Elimination has expected regret

$$O\left(\sqrt{TK \ln |\Pi|}\right).$$

A key lemma: For any set of policies $\Pi$ and any distribution over $x$, step 1 is possible.
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Proof: Consider the game:

$$\min_P \max_Q E_{\pi \sim Q} E_x \left(\frac{1}{1 - K \mu_t} \Pr_{\pi' \sim P}(\pi(x) = \pi'(x)) + \mu_t\right)$$
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Minimax magic!

$$= \max_Q \min_P E_{\pi \sim Q} E_x \left(\frac{1}{1 - K \mu t} \Pr_{\pi' \sim P}(\pi(x) = \pi'(x)) + \mu t\right)$$
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Let $P = Q$

$$\leq \max_Q E_{\pi \sim Q} E_x \left(1 - K \mu t \right) \Pr_{\pi' \sim Q}(\pi(x) = \pi'(x)) + \mu t$$
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For all sets of policies $\Pi$, for all distributions $D(x, \tilde{r})$, if the world is IID w.r.t. $D$, with high probability Policy Elimination has expected regret

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A key lemma: For any set of policies $\Pi$ and any distribution over $x$, step 1 is possible.

Proof: Consider the game:

$$\min_P \max_Q \mathbb{E}_{x} \left( \frac{1}{(1-K\mu_t) \Pr_{\pi' \sim P}(\pi(x)=\pi'(x)) + \mu_t} \right)$$

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Let $P = Q$

$$\leq \max_Q \mathbb{E}_{x} \sum_a \frac{\Pr_{\pi \sim Q}(\pi(x)=a)}{(1-K\mu_t) \Pr_{\pi' \sim Q}(\pi'(x)=a) + \mu_t}$$

Linearity of Expectation

$$= \max_Q \mathbb{E}_{x} \sum_a \frac{\Pr_{\pi \sim Q}(\pi(x)=a)}{(1-K\mu_t) \Pr_{\pi' \sim Q}(\pi'(x)=a) + \mu_t}$$
Another Use of minimax
[DHKKLRZ11]

Randomized_UCB

Let $\mu_t = \text{minimum action probability.}$
$\Delta_t(\pi) = \max_{\pi'} \eta_t(\pi') - \eta_t(\pi)$
For each $t = 1, 2, \ldots$

1. Choose distribution $P$ over $\Pi$ minimizing $E_{\pi \sim P}[\Delta_t(\pi)]$ s.t. $\forall \pi$:

   $E_{x \sim h_t} \left[ \frac{1}{(1 - K\mu_t) \Pr_{\pi' \sim P}(\pi'(x) = \pi(x)) + \mu_t} \right]$

   $\leq \max\{2K, Ct(\Delta_t(\pi))^2\}$

   using oracle learning algorithm(*).

2. observe $x_t$

3. Let $p_t(a) = (1 - K\mu_t) \Pr_{\pi' \sim P_t}(\pi(x) = a) + \mu_t$

4. Choose $a_t \sim p_t$ and observe reward $r_t$

(*) Much complexity hidden here.
Another Use of minimax

[DHKKLRZ11]

Randomized_UCB

Let $\mu_t = \text{minimum action probability.}$
$\Delta_t(\pi) = \text{empirical regret}$
For each $t = 1, 2, \ldots$

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[DHKKLRZ11]

Randomized_UCB

Let $\mu_t =$ minimum action probability.
$\Delta_t(\pi) =$ empirical regret
For each $t = 1, 2, \ldots$

1. Use oracle learning algorithm to find a sparse distribution $P$ over $\Pi$ inducing large probability on all good policy’s actions and possibly small probability on bad policy’s actions.

2. observe $x_t$

3. Let $p_t(a) = (1 - K\mu_t) \Pr_{\pi \sim P_t}(\pi(x) = a) + \mu_t$

4. Choose $a_t \sim p_t$ and observe reward $r_t$
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For all sets of policies $\Pi$, for all distributions $D(x, \bar{r})$, if the world is IID w.r.t. $D$, with high probability Randomized UCB has expected regret

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And: Given an cost sensitive optimization oracle for $\Pi$, Randomized UCB runs in time $\text{Poly}(t, K, \log |\Pi|)$.
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And: Given an cost sensitive optimization oracle for $\Pi$, Randomized UCB runs in time $\text{Poly}(t, K, \log |\Pi|)!$

Uses ellipsoid algorithm for convex programming. First ever general $\text{Poly}(\log |\Pi|)$ algorithm for contextual bandits.
Final Thoughts and pointers

We can be aggressive in other ways as well. For example, if rewards are delayed, regret is additive rather than multiplicative in the delay.


Further Contextual Bandit discussion: http://hunch.net/