Learning Graphs with Queries

Lev Reyzin

Clique talk

Fall 2006

Some Papers

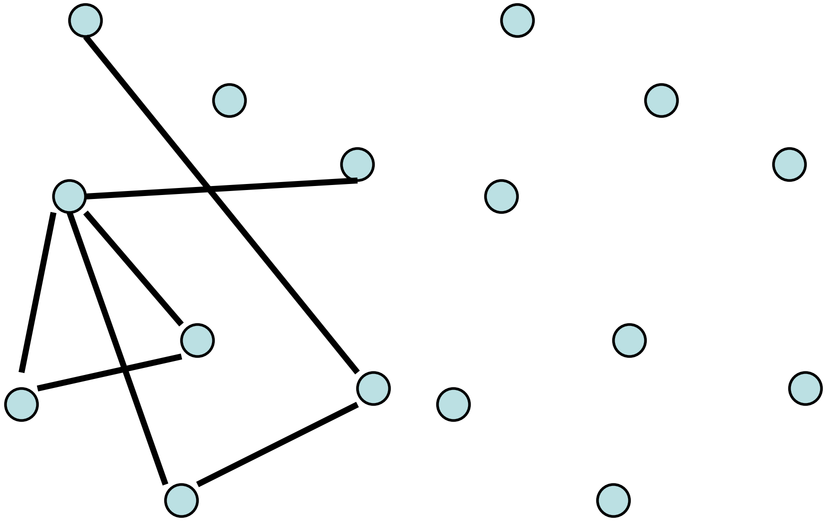
- Angluin, Chen**. Learning a Hidden Graph Using *O*(log *n*) Queries Per Edge.** (COLT '04)

- Bouvel, Grebinski, Kucherov. **Combinatorial Search on Graphs Motivated by Bioinformatics Applications: A Brief Survey** (WG '05)

- Reyzin, Srivastava. **A Survey of Graph Learning with Queries** (Manuscript '06)

- Hein. **An Optimal Algorithm to Reconstruct Trees from Additive Distance Data.** (Journal of Math Bio '89)

Hidden Graphs

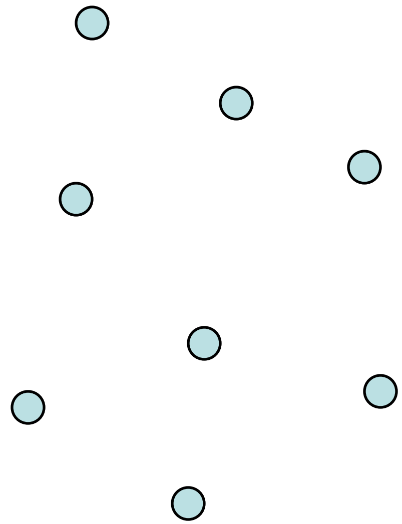


Oracle at Delphi

Queries

Edge
query pair of vertices in graph to detect presence of edge
(social networks)

Edge Counting
query set of vertices in graph for number of edges
(DNA sequencing)

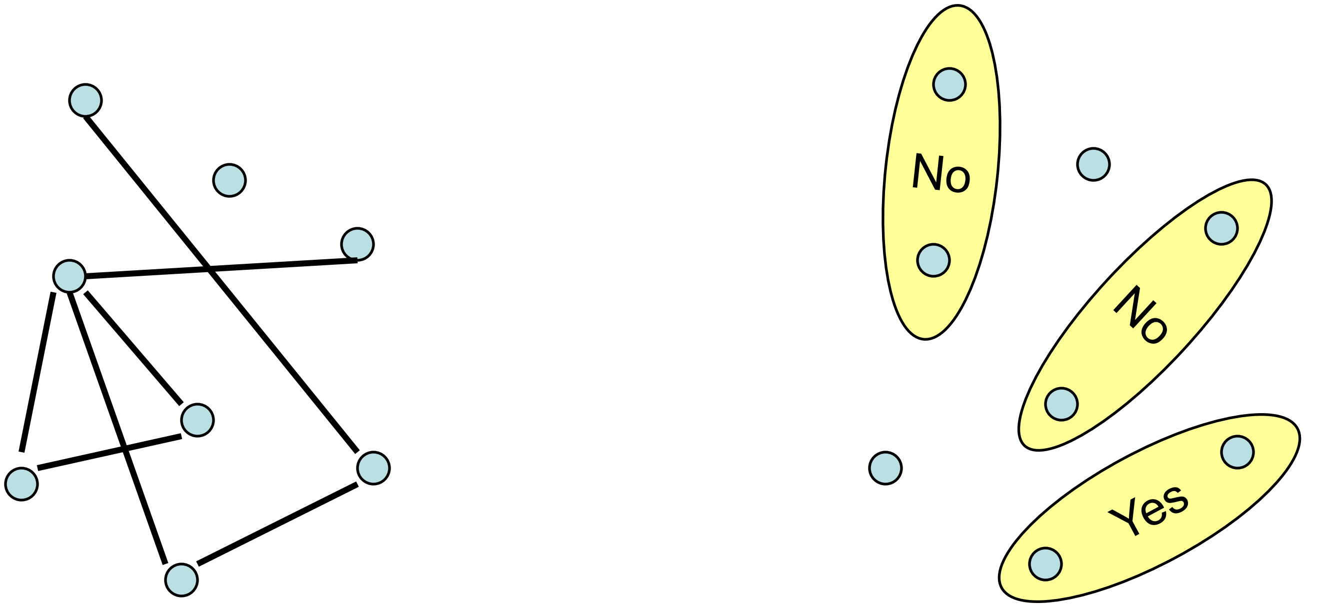


Edge Detection
query set of vertices in graph for presence of edges
(chemical reactions)

Shortest Path
query two vertices in graph for shortest path length between them
(evolutionary trees)

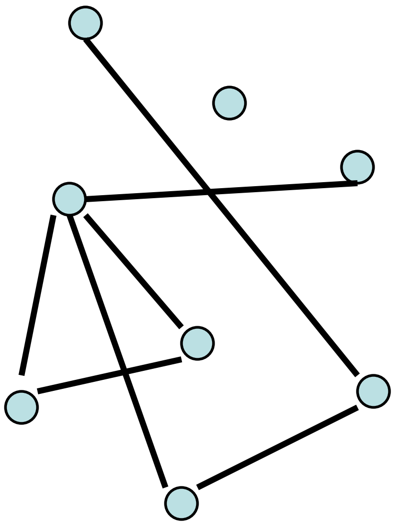
Connectedness
query pair of vertices in graph to discover if they're in same component
(electrical networks)

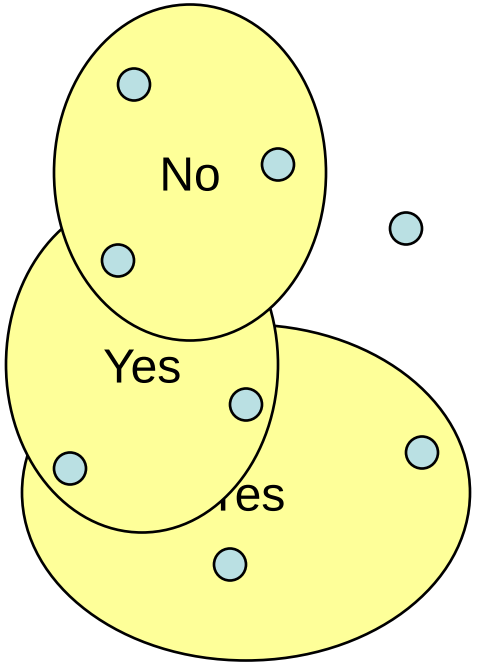
Edge Queries



Edge Query: query pair of vertices in graph to detect presence of edge

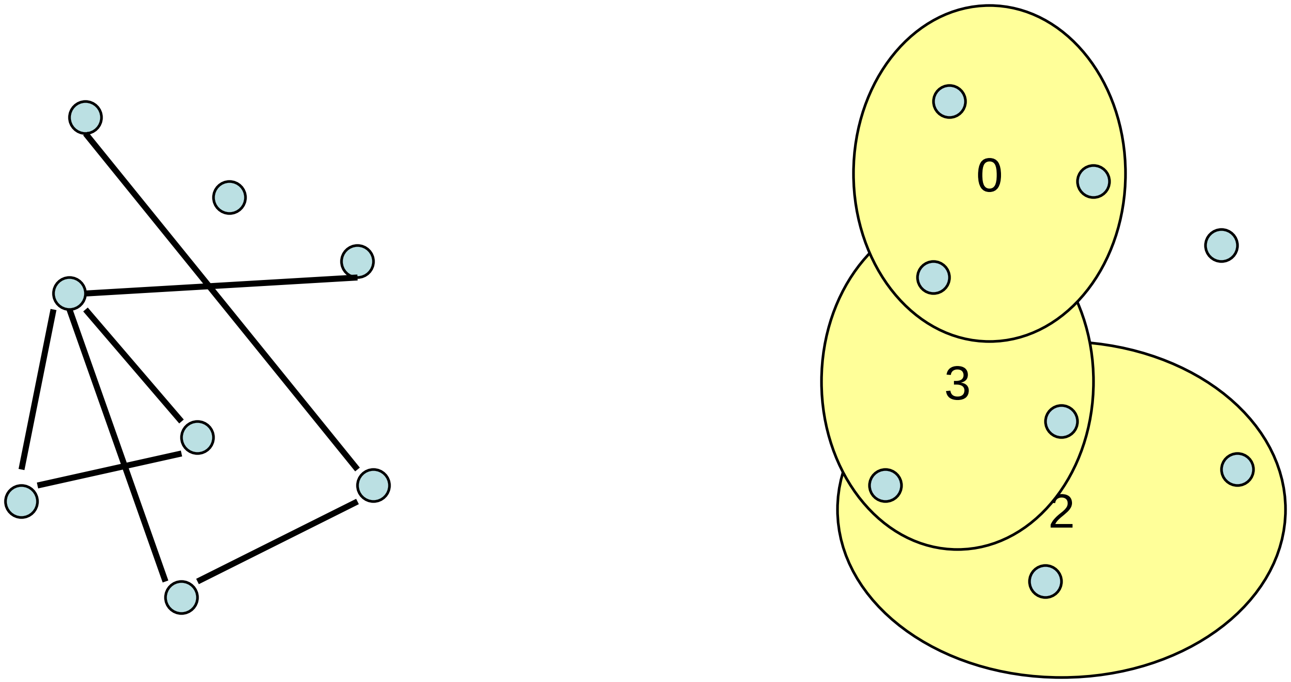
Edge Detection Queries





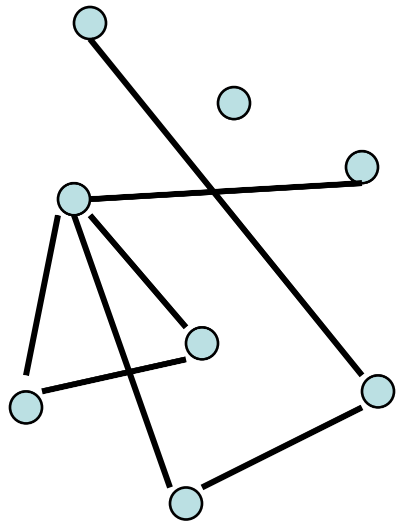
Edge Detection Query: query a set of vertices in graph for presence of an edge

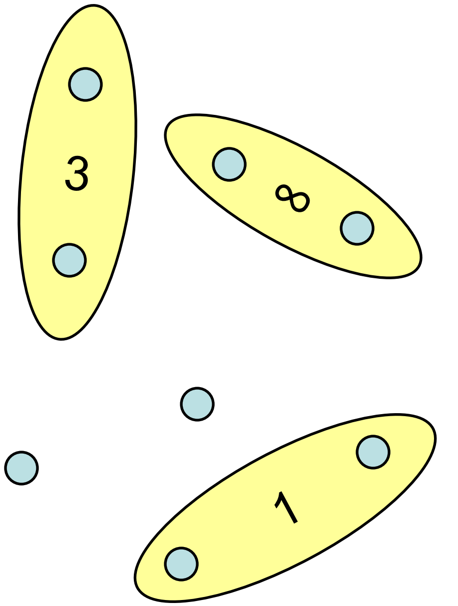
Edge Counting Queries



Edge Counting Query: query a set of vertices in a graph for number of edges in subgraph induced by the vertices

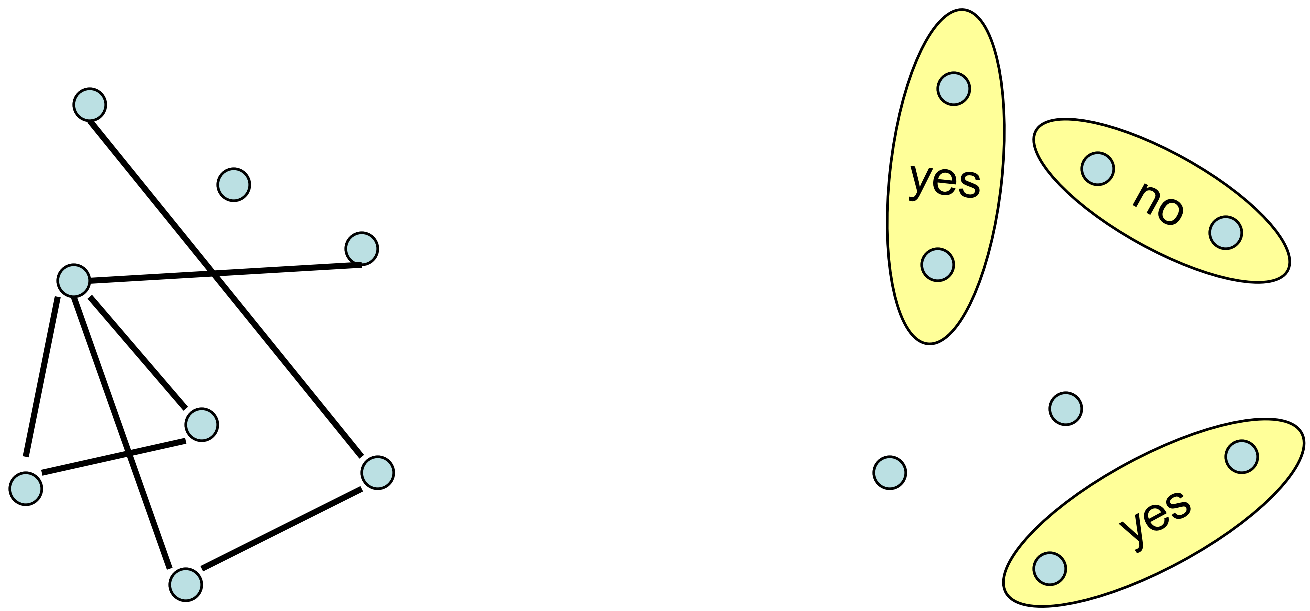
Shortest Path Queries





Shortest Path: query two vertices in graph for shortest path length between them

Connectedness Queries



Connectedness Queries: query pair of vertices in graph to discover if they’re in same component

Relative Power of Queries

upper bounds

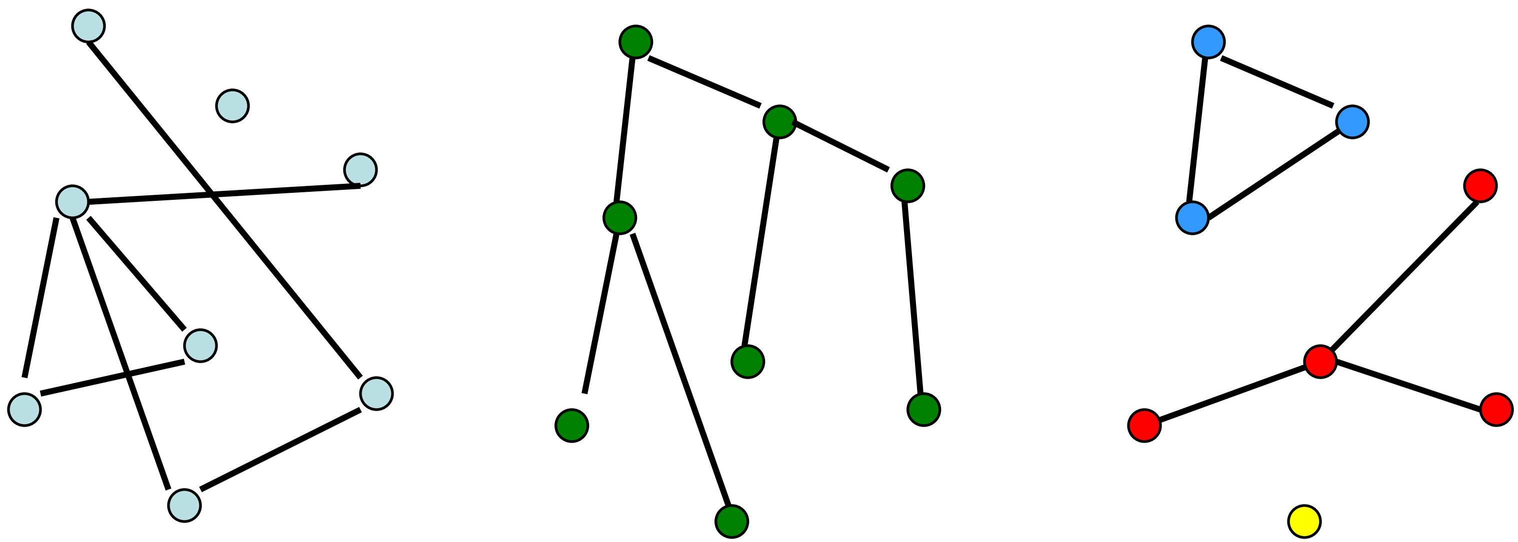
$$\mathbf{E} \leq \mathbf{ED} \leq \mathbf{EC}$$

$$\mathbf{E} \leq \mathbf{SP}$$

$$\mathbf{C} \leq \mathbf{SP}$$

lower bounds

Target Classes of Graphs



general graphs trees partitions

Results Summary

n = number of nodes, |E| = number of edges,
k = number of connected components, d = maximum degree

Query	partition	graph	tree
E	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n^2)$
ED	$\Theta(n^2)$	$\Theta(\lvert E\rvert \lg n), \Theta(n^2)$	$\Theta(n \lg n)$
EC	$O(n \lg n)$ $\Omega(n)$	$O(\lvert E\rvert \lg n), O(\frac{n^2}{\lg n}), O(dn)$ $\Omega(dn)$	$\Theta(n)$
SP	$\Theta(nk)$	$\Theta(n^2)$	$\Theta(n^2), \Theta(dn \lg n)$
C	$\Theta(nk)$	not possible	not possible

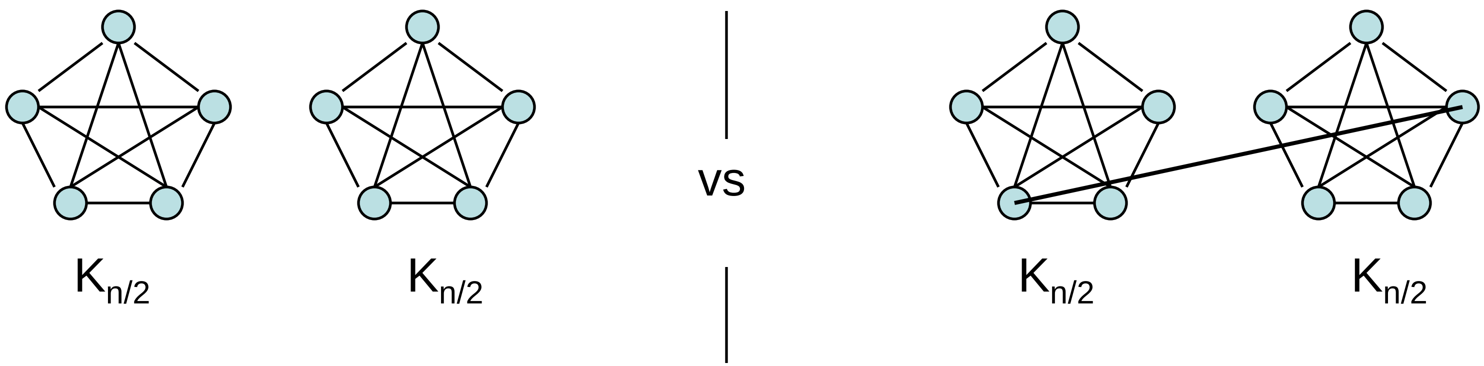
E = Edge Query, ED = Edge Detection Query, EC = Edge Counting Query,
SP = Shortest Path Query, C = Connectedness Query

references omitted

Learning Partitions with ED

- Upper bound: $O(n^2)$ – trivial. query all pairs of vertices.
- Lower bound: $\Omega(n^2)$

Distinguishing



requires $\Omega(n^2)$ by adversarial argument

Results Summary

n = number of nodes, |E| = number of edges,
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SP	$\Theta(nk)$	$\Theta(n^2)$	$\Theta(n^2), \Theta(dn \lg n)$
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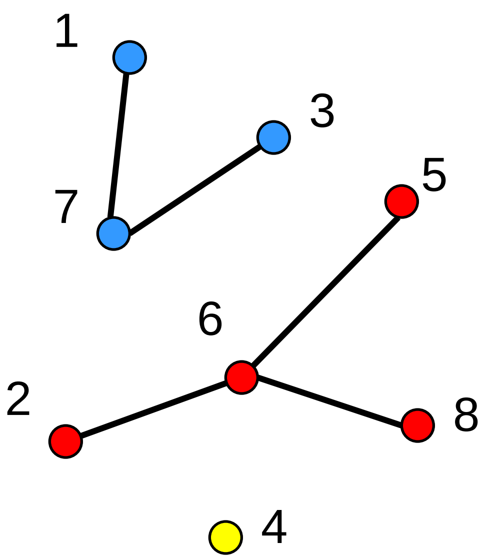
Learning Partitions with C

- Upper bound: O(nk)

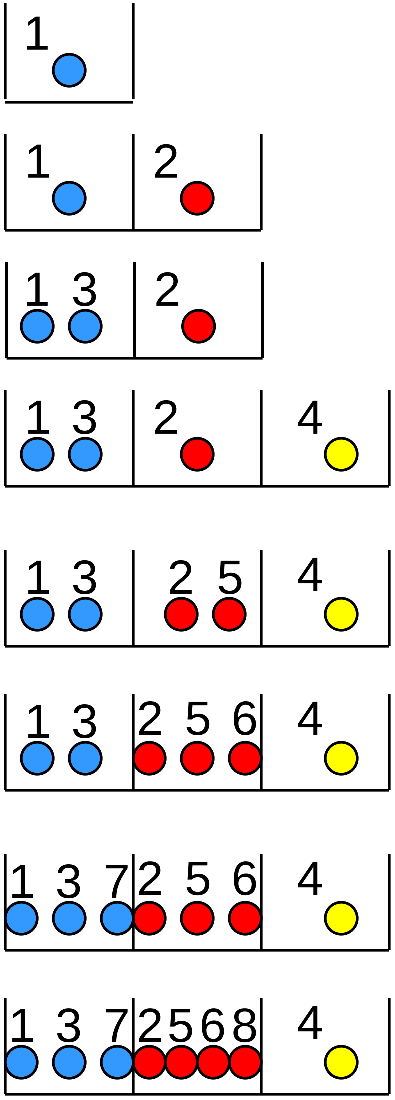
  algorithm

  - Step 1: Place v_1 in its own component
  - Step i > 1: Query $C(v_i,v_w)$ for items v_w from each existing component; if a "yes" is encountered, place v_i in the corresponding component and continue to step i+1. Otherwise place v_i in its own component.

An Example



partitions



Learning Partitions with C

- Upper bound: O(nk)
  - correctness: trivial
  - running time: For complexity, note that there at most k components at any step (since there are at most k components at phase n and components are never destroyed); hence n vertices take at most nk queries.

Back to Our Table

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Learning Partitions with EC

- Lower bound: $\Omega(n)$

  information theoretic argument:

  – The number of partitions of an element set is given by the Bell number B_n.

  – $\lg(B_n) = n \lg n$ (de Bruijn '81)

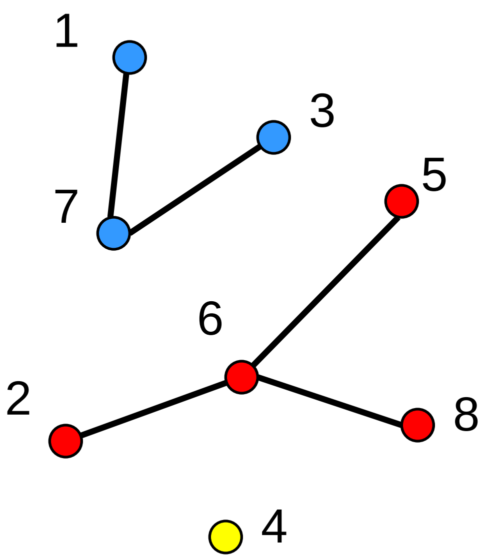
  – each EC query gives $\lg(C(n,2)) = \lg n$ bits.

  – we need $\Omega((n \lg n) / (\lg n)) = \Omega(n)$

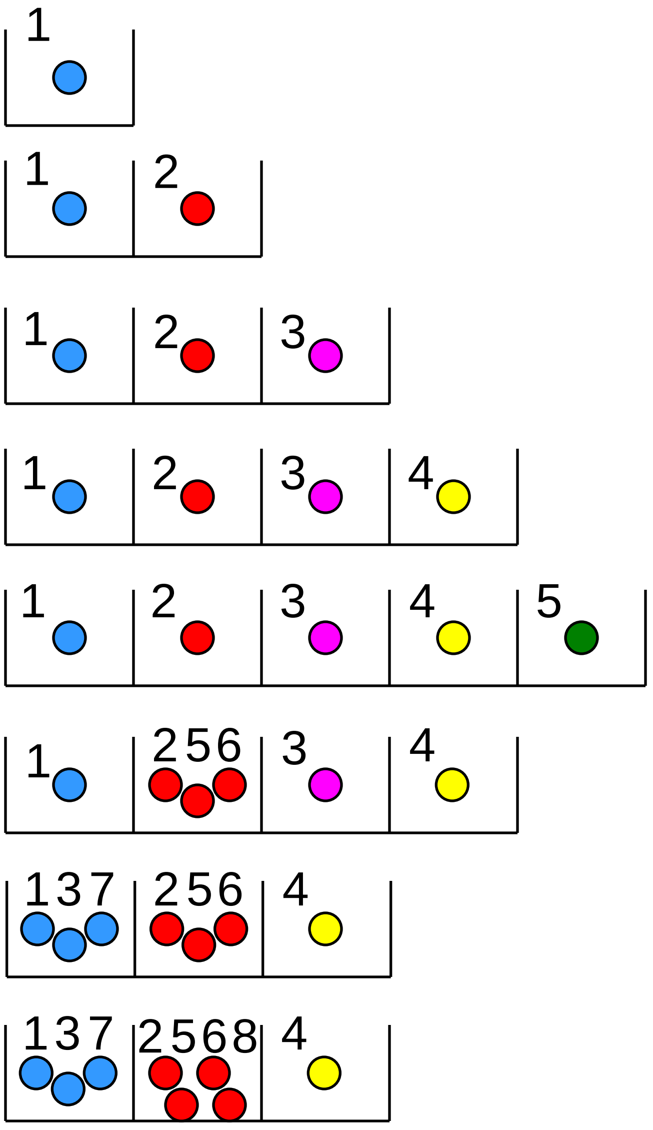
An Algorithm for the Upper Bound

- Upper bound: $O(n \log n)$
  - Phase 1: set $C = \{c_1\}$ with $c_1=1$
  - Phase i : let $v = (v_{i+1})$ query $EC(C+v)$
    - if $EC(C+v)=EC(C)$ add a new component $c=v$ to C
    - else split C into roughly equal halves C_1 and C_2 and query $EC(C_1+v)$,$EC(C_2+v)$. Recurse until $EC(\{c_j\}+v) > EC(c_j)$ for a single component c_j. Call c_j a live component. Repeat recursively on C/c_j until all live components are found. Merge them and v into 1 component in C.

An Example



partitions



Proof Sketch

- Correctness of the algorithm is simple by induction on the phase. C contains the components of G[1 . . . i] at end of phase i
- Running time is bounded by O(n lg n) because we do O(lg n) queries to find each live component. But each time we find a live component, the number of components in our set decreases by 1. Since we can have at most n partitions, the total running time is bounded by O(n lg n)

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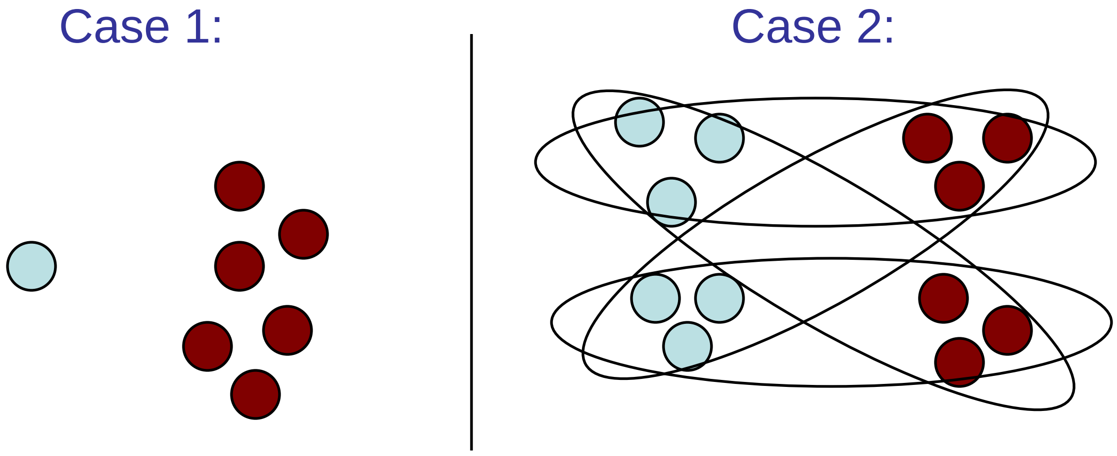
Learning Graphs with ED
(Angluin, Chen COLT '04)

- Upper bound: O(|E| log n)
  - Lemma 1: if S_1 and S_2 are two non-empty independent sets of vertices in G. We can find s edges between S_1 and S_2 in O(s log n)



Learning Graphs w/ ED - continued
(Angluin, Chen COLT '04)

- Upper bound: $O(|E| \log n)$
  - Fact: A graph with m edges can be $O(m)^{1/2}$ colored
    - we can collapse pairs of vertices not joined by an edge, until we get a clique of m edges. It can be O(t)-colored and has $O(t^2)$ vertices.
  - Lemma 2: if S_1 and S_2 ($|S_1| < |S_2|$ are two non-empty sets of vertices in G (w/ s_1 and s_2 edges respectively). We can find s edges between S_1 and S_2 in $O(s*\log|S_2|+s_1+s_2)$
    - we color both S_1 and S_2 with $(s_1)^{1/2}$ and $(s_2)^{1/2}$ colors
    - for each pair of color classes in S_1 and S_2, we query the union
    - recurse

Learning Graphs w/ ED - continued
(Angluin, Chen COLT '04)

- Upper bound: O(|E| log n)
  - The Algorithm:
    - 1) If |V| = 2, mark pair of vertices as edge andreturn
    - 2) Divide V into halves S1 and S2. Ask $ED(S_1)$ and $ED(S_1)$
    - 3) Recursively solve the problem for S_i if $ED(S_i)=1$
    - 4) Using lemma 2, find edges between S_1 and S_2

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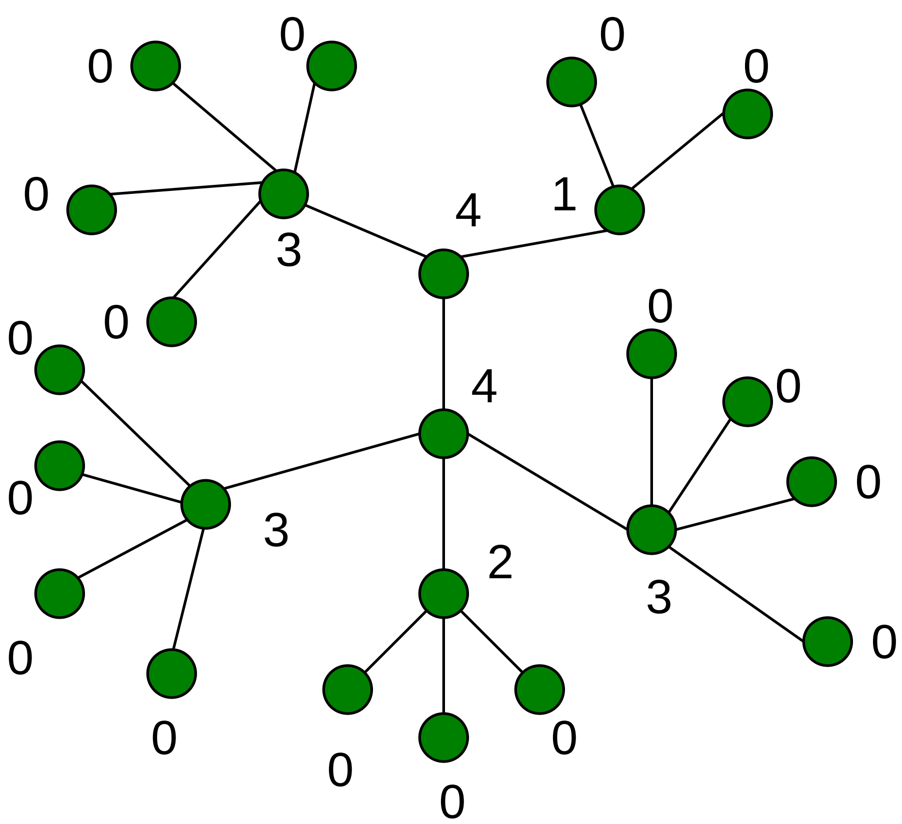
Learning Trees with SP Queries (Hein, '89)

- Upper bound: O(d n log n)
  - Proof intuition:



Open Problems and Future Work

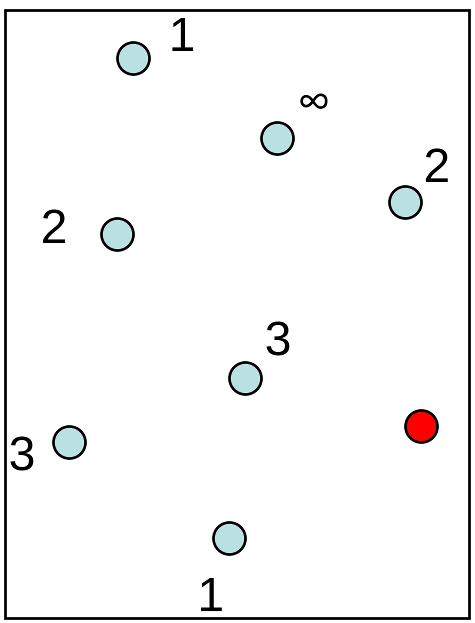
Close the gap for partition with EC queries

Consider other queries: traceroute queries, beacon queries,…

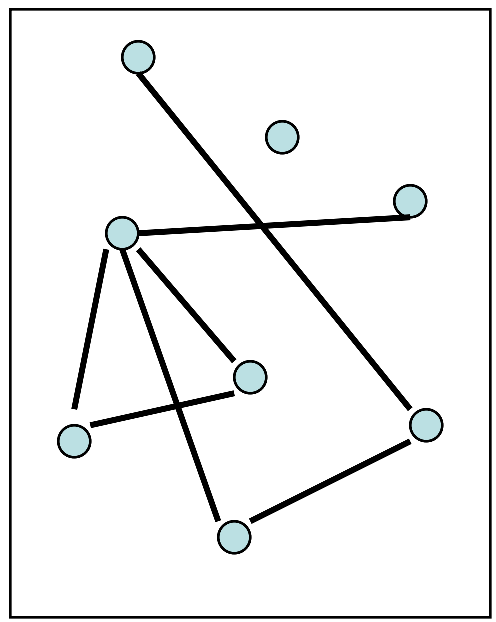
Graph Verification. Given a result from a certain class, how hard is it to verify. Very interesting: verifying general graphs with EC queries.

Beacon and Traceroute Queries

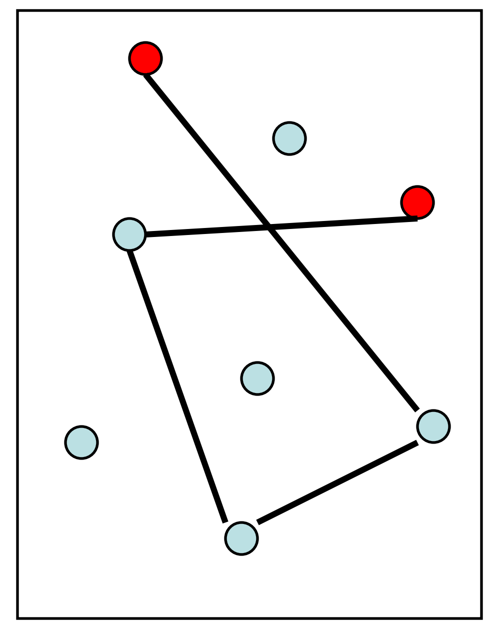
Beacon



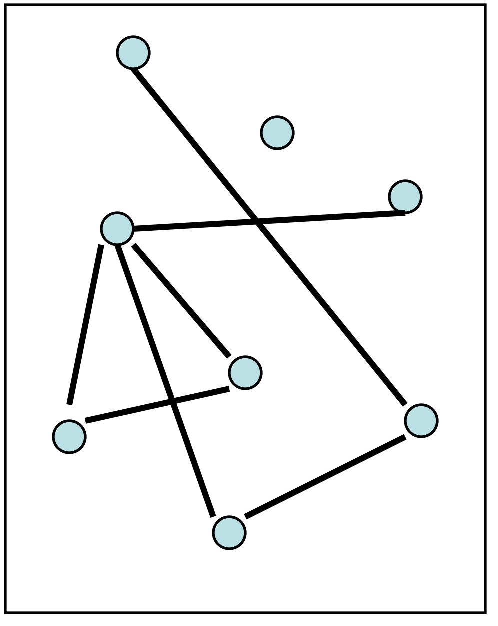
Target



Traceroute



Verification



Thank You

Questions?